DESIGN AND TEST OF AN HYPERBOLIC ELBOW FLOW METER

BY

JULIUS M. NAIMAN

ARMOUR INSTITUTE OF TECHNOLOGY

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DESIGN AND TEST OF AN HYPERBOLIC ELBOW FLOW METER

A THESIS

PRESENTED BY

JULIUS M. NAIMAN

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APPROVED

George F. Gebhardt
Professor of Mechanical Engineering

H. M. Raymond
Dean of Engineering Studies

L. C. Monin
Dean of Cultural Studies

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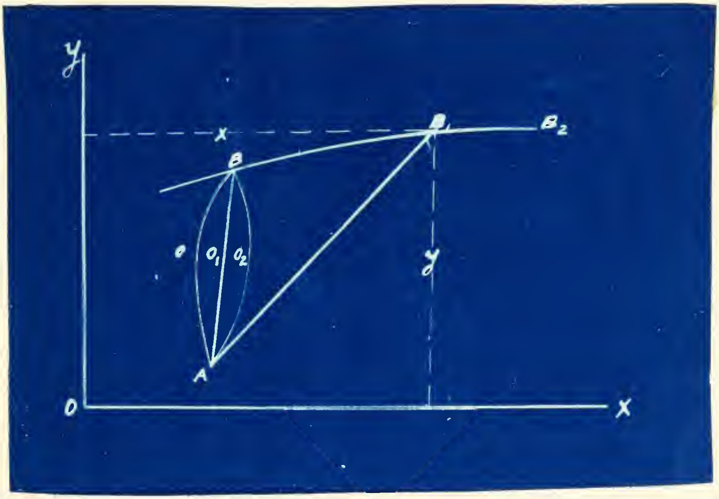
APPENDIX A

ELEMENTARY THEORY of PLANE HYPERBOLIC STREAM LINE FLOW

INCOMPRESSIBLE FLUIDS

1. By flux across any curve connecting two points is meant the volume of liquid crossing the right cylindrical surface enveloping the curve and included between two parallel plane bases unit distance apart.



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2. Thus, in Fig. 1a, the volume of liquid, crossing the curve AOB connecting the two points A and B and passing between two planes parallel to the paper and unit distance apart, is called the flux passing across AOB Evidently the flux passing through AOB must also pass through AO_1B and AO_2B, otherwise, in time there would have to be either a decrease or an increase of the mass of fluid included in the spaces $AOBO_1$ and AO_1BO_2. But if the motion is steady, there evidently cannot be any such changes in the mass of fluid with time. Therefore, we conclude that the same flux passes across any curve connecting the points A and B; also, if A is fixed, it is evident that the value of the flux depends only upon the position (x,y) of B and is independent of the nature of the curve connecting A and B; i.e.,

$$\text{Flux} = \phi(x,y), \text{ only.}$$

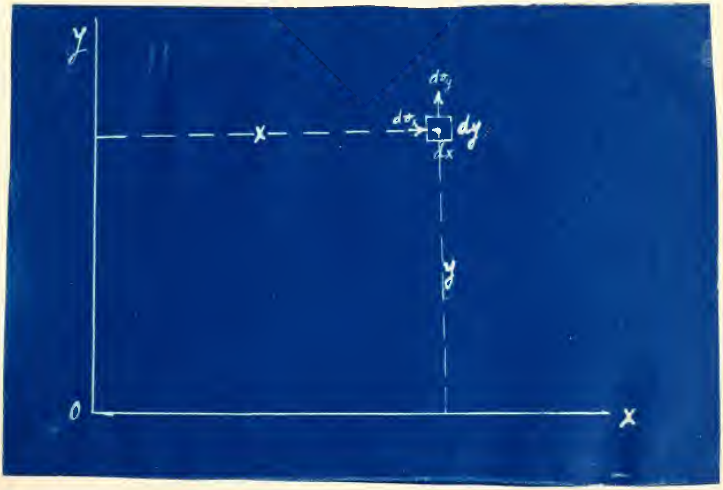
3. If now another point, B_1, very close to B, is found such that the flux passing across AB_1 is the same as that passing across AB, there cannot be any flux passing across the curve BB_1; i.e., the fluid must flow along the curve BB_1 without crossing the latter. This path BB_1 is commonly known as a stream line.

4. Since it appears that, on the one

hand the flux passing through the lines A,B, A,B_1 etc. depends upon the locations (x,y) of the points B,B_1, etc., while, on the other hand, this same flux does not vary if the points B,B_1 etc. happen to lie on a stream line BB_1B_2, whose equation, let us say, is $-xy = C$, it is evident that the flux must depend in some way upon a common property of B,B_1 etc., or upon the equation, $-xy = C$, of the stream line BB_1B_2. Thus, for the assumed hyperbolic stream lines, the flux must be some function of $-xy = C$ i.e.,

$$\text{Flux} = f\,(-xy) \qquad (1a)$$

Since $-xy = C$ remains constant as we move along the hyperbolic stream line curve, BB_1B_2, any function of $-xy$, e.g.,



$f(-xy)$ or the flux must also remain constant, as is required.

5. If now, we analyze the flux passing through an infinitessimal volume, of area dxdy (Fig.2a) and of unit height, we note, that if the horizontal velocity component, v_x, brings in more flux than it removes, the vertical velocity component, v_y, must remove more flux than it brings in, in order that there should be no net change of mass in the infinitessimal volume dxdy, with time; if the first change of flux is an increase, the second change must be a decrease. WE have then,

$$\partial^2 f = \partial v_x \partial y \qquad (2a)$$

$$\partial^2 f = -\partial v_y \partial x \qquad (3a)$$

or,

$$\frac{\partial^2 f}{\partial y} = \partial\left(\frac{\partial f}{\partial y}\right) = \partial v_x \qquad (2a)$$

or,

$$\frac{\partial^2 f}{\partial x} = \frac{\partial\left(\partial f\right)}{\partial x} = -\partial v_y \qquad (3a)$$

and finally,

$$\frac{\partial f}{\partial y} = v_x \quad \text{or} \quad v_x = \frac{\partial f}{\partial y} \qquad (2a)$$

$$\frac{\partial f}{\partial x} = -v_y \quad \text{or} \quad v_y = -\frac{\partial f}{\partial x} \qquad (3a)$$

6. There is thus seen to be a definite relation between flux and velocity components. For frictionless flow, there exists in addition what is known as a velocity potential, (see Lamb's Hydrodynamics; p. 69-70 and preceding chapters) which is characterized by the following equations:

$$v_x = - \frac{\partial \phi}{\partial x} \qquad (4a)$$

$$v_y = - \frac{\partial \phi}{\partial y} \qquad (5a)$$

7. Physically this means that the fluid moves without rotation or that all the component angular velocities are zero. From the general equations for the component angular velocities, it can be shown by equating these velocities to zero that there is a function, which is characterized by equation (4a) and (5a) as given above. This function is called velocity potential, because of the similarity of its analytical expression to that for electrical potential, as explained in Lamb's Hydrodynamics.

8. From equations (4a and 5a) we get by differentiation,

$$\frac{\partial v_x}{\partial y} = \frac{-\partial^2 \phi}{\partial x \partial y} \qquad (4a)$$

$$\frac{\partial v_y}{\partial x} = - \frac{\partial^2 \phi}{\partial y \partial x} \qquad (5a)$$

or by combination

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \qquad (6a)$$

But from equation (2a and 3a)

we get,

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 f}{\partial y^2} \qquad (2a)$$

$$\frac{\partial v_y}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \qquad (3a)$$

Therefore, remembering equation (6a) we may write

$$\frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 f}{\partial x^2} \qquad (7a)$$

9(Now, is we express the flux function f (-xy), as an infinite series, we have

$$f(xy) = A - B\,x\,y + C\,x^2y^2 - D\,x^3y^3 + \dots$$

from which we obtain by differentiation,

$$\frac{\partial f}{\partial x} = -By + 2Cxy^2 - 3Dx^2y^3 + \dots$$

$$\frac{\partial^2 f}{\partial x^2} = 2cy^2 - 6Dxy^3 + \dots = 2y^2(C - 3Dxy + .$$

$$\frac{\partial f}{\partial y} = -Bx + 2Cx^2y - 3Dx^3y^2 + \dots$$

$$\frac{\partial^2 f}{\partial y^2} = 2Cx^2 - 6Dx^3y + \dots = 2x^2(C - 3Dx$$

Comparing the expressions for $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x^2}$ and remembering equation (7a), we get,

$$2x^2(C-3Dxy+.) = -2y^2(C-3Dxy+\dots) \qquad (9a)$$

or

$$x^2 = -y^2 \qquad (10a)$$

for all values of x and y. The last condition is evidently impossible. This fact indicates that the terms cancelled in equation (9a) are equal to zero i.e.,

$$C - 3Dxy + \dots\dots\dots\dots = 0$$

for all values of xy, which in turn can only mean that

$$C = 0, \ D = 0, \ \text{etc.},$$

or that for frictionless flow, the nature of the flux function is,

$$f(xy) = A - Bxy \qquad (11a)$$

all the other terms of the series disappearing.

10. Substituting the last expression for the flux function into the velocity equations (2"a and 3"a) we get,

$$v_x = \frac{\partial f}{\partial y} = -Bx \qquad (12a)$$

$$v_y = -\frac{\partial f}{\partial x} = By \qquad (13a)$$

Also the resultant velocity, v can now be expressed as,

$$v = \sqrt{V_x^2 + V_y^2} = B\sqrt{x^2 + y^2} = \qquad (14a)$$

$$= Br$$

for all values of x and y. The last condition is evidently impossible. This fact indicates that the terms cancelled in equation (9a) are equal to zero i.e.,

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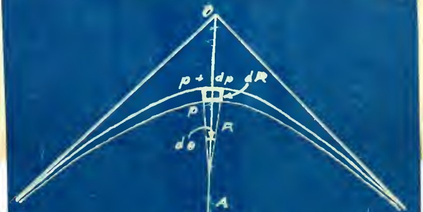


FIG. 3A

11. If we next analyze the effect of the centrifugal forces acting along the line of symmetry, OA, of the hyperbolic stream lines of Fig.3A, we note that the centrifugal force acting upon the infinitessimal fluid particle, of vertical height H, and horizontal cross-section,

$a = R\,d\,\theta\,dR$, is equal to,

$$\text{Centrifugal force} = \frac{m\,v^2}{R}$$

$$= \frac{D}{g}\,\frac{aHv^2}{R} = \frac{D}{g}\,\frac{Rd\theta dRHv^2}{R} = \frac{D}{g}\,Hv^2d\theta dR \qquad (15a)$$

This force is balanced by a centripetal force in the form of pressure variation along the line of symmetry, the value of which is,

$$A\,dp = HRd\theta\,dp \qquad (16a)$$

Since these two forces must be equal and opposite, we have,

$$\frac{D}{g}\,Hv^2d\theta dR = -HRd\theta dp \quad (17a)$$

or

$$\frac{dp}{dR} = \frac{-Dv^2}{gR} \qquad (18a)$$

12. The term R in the last equation represents the radius of curvature of any hyperbola at its intersection with the line of symmetry. This happens to be numerically equal to the shortest

radius vector of the corresponding hyperbola.

Thus,

$$R = \frac{\left[1 - \left(\frac{\partial y}{\partial x}\right)^2\right]^{3/2}}{\frac{\partial^2 y}{\partial x^2}} \qquad (19a)$$

and since,

$$xy = C = -\frac{R^2}{2}$$

$$\frac{dy}{dx} = -\frac{C}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}$$

$$\frac{dy^2}{dx} = \frac{2C}{x^3} = \frac{2xy}{x^3} = \frac{2y}{x^2}$$

we have,

$$R = \frac{\left(1 + \frac{y^2}{x^2}\right)^{3/2}}{\frac{2y}{x^2}} = \frac{x^2\left(\frac{x^2+y^2}{x^2}\right)^{3/2}}{2y} =$$

$$= \frac{x^2\left(\frac{R^2}{x^2}\right)^{3/2}}{2y} = \frac{x^2 R^3}{2x^3 y} =$$

$$= \frac{+R^3}{2xy} = \frac{R^3}{-R^2} = -R \qquad (20a)$$

as was stated.

13. An inspection of equation (18a) and (14a) shows that we may write.

$$\frac{dp}{dR} = -\frac{D}{g}\frac{V^2}{R} = -\frac{D}{g}\frac{B^2R^2}{R}$$

$$= \frac{D}{g} R B^2 \qquad (21a)$$

since r = R, along the line of symmetry.

14. Integrating equation (21a) with respect to R, we get

$$p = p_0 - \frac{DB^2}{2g} R^2 \qquad (22a)$$

The difference of pressure between the inside and outside curves of a hyperbolic elbow at the points of shortest radii vectors R_1, R_2 may now found, Thus,

$$p_1 - p_2 = \frac{D}{2g} B^2 \left(R_2^2 - R_1^2\right) \qquad (23a)$$

15. Again from equation (14a) the expression for mean velocity is

$$v_a = \frac{\int_{R_1}^{R_2} V\,dR}{R_2 - R_1} = \frac{\int_{R_1}^{R_2} BR\,dR}{R_2 - R_1} =$$

$$= B\frac{\left(R_2^2 - R_1^2\right)}{2\left(R_2 - R_1\right)} = B\frac{\left(R_2 + R_1\right)}{2} \qquad (24a)$$

16. Comparing equations (23a) and (24a), we get,

$$v_a = B\frac{\left(R_2 + R_1\right)}{2} = \frac{\sqrt{\frac{2g\left(p_1 - p_2\right)}{D}}}{\sqrt{R_2^2 - R_1^2}} \quad \frac{\left(R_2 + R_1\right)}{2} =$$

$$= \frac{R_2 + R_1}{2\sqrt{R_2^2 - R_1^2}} \sqrt{2gH} \quad =$$

$$= \sqrt{\frac{(R_2 + R_1)^2}{4(R_2^2 - R_1^2)}} \sqrt{2gH} \quad = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}} \sqrt{2gH}$$

$$= K\sqrt{2gH} \qquad (25a)$$

From which the expression for the flow meter coefficient, K, in the case of frictionless flow, is,

$$K = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}} \qquad (26a)$$

17. For flow with friction, the expression for flux takes the more general form,

$$f(xy) = A - Bxy + Cx^2y^2 - Dx^3y^3 + \ldots$$

(see eq. 8a and compare with eq.11a)

18. Therefore, the expressions for the velocity components now become,

$$\begin{aligned} v_x = \frac{\partial f}{\partial y} &= -Bx + 2Cx^2y - 3Dx^3y^2 + \ldots = \\ &= -x\left(B - 2Cxy + 3Dx^2y^2 - \ldots\right) = \\ &= -x\,F(-xy) \end{aligned}$$

(27a)

$$V_y = -\frac{\partial f}{\partial x} = -\left(-By + 2Cxy^2 - 3Dx^2y^3 + \dots\right) =$$

$$= y\left(B - 2Cxy + 3Dx^2y^2 - \dots\right) =$$

$$= y\,F(-xy)$$

(28a)

Similarly, the expression for resultant velocity becomes,

$$v = \sqrt{V_x^2 + V_y^2} = F(-xy)\sqrt{x^2 + y^2} =$$

$$= rF\ (-xy) \qquad (29a)$$

In these equations, F is an unknown function of -xy depending in some way upon internal friction. When internal friction is negligible, this function becomes,

$$F\ (-xy) = B$$

where B = constant.

(Compare Eq. 12a,13a,14a with eq. 27a,28a,29a,)

19. The equation for pressure distribution viz.,

$$\frac{dp}{dR} = -\frac{D\ v^2}{g\ R} \qquad \text{(see eq. 18a)}$$

holds for flow with friction just the same as it did for flow without friction since the centrifugal forces must still be balanced by a proper pressure distribu-

tion along the line of symmetry. However, since the law of velocity distribution has been changed to

$$V = R\,F \qquad \text{(See eq. 29a)}$$

eq. 18 takes the form,

$$\frac{dp}{dR} = -\frac{D}{g}\,RF^2 \qquad (30a)$$

(Compare with eq.21a)

where the same unknown function F is involved as in the expression for velocity.

20. By analyzing experimentally the pressure distribution along the line of symmetry of the hyperbolic elbow, as described in paper proper, the nature of the "F" function was found to be in the case of water either,

$$F\,(R) = B\,(1 + A_1 R) \qquad (30a)$$

(Compare with eq.38)

or

$$F\,(R) = C\,(R)^{\frac{1}{n}} \qquad (31a)$$

(Compare with eq. 37)

21. Therefore, using the first expression for the F function we get the following changes in eq. 22a to 27a inclusive:

$$p = \int -\frac{D}{g}\,RF^2 dR =$$

$$= p_0 - \frac{DB^2}{g}\int (R + 2A_1R^2 + A_1^2R^3)\,dR$$

$$= p_0 - \frac{DB^2}{g}\left(\frac{R^2}{2} + \frac{2}{3}A_1R^3 + \frac{A_1^2}{4}R^4\right) \qquad (32a)$$

$$p_1 - p_2 = \frac{DB^2}{g}\left[\frac{R_2^2-R_1^2}{2} + \frac{2}{3}A_1\left(R_2^3-R_1^3\right) + \frac{A_1^4}{4}\left(R_2^4-R_1^4\right)\right] \quad (33a)$$

$$V_a = \frac{\int_{R_1}^{R_2} v\,dR}{R_2 - R_1} = B\,\frac{\int_{R_1}^{R_2} R\,(1+A_1R)\,dR}{R_2 - R_1}$$

$$= B\left[\frac{R_2^2 - R_1^2}{2\,(R_2 - R_1)} + \frac{A_1\,(R_2^3 - R_1^3)}{3\,(R_2 - R_1)}\right]$$

$$= B\left[\frac{R_2 + R_1}{2} + \frac{A_1}{3}\left(R_2^2 + R_2R_1 + R_1^2\right)\right] \quad (34a)$$

Also comparing the new equations for average velocity and difference of pressure, we get,

$$V_a = B\left[\frac{R_2+R_1}{2} + \frac{A_1}{3}\left(R_2^2 + R_2R_1 + R_1^2\right)\right] =$$

$$= \frac{\sqrt{\frac{2g\,(p_1-p_2)}{D}}\left[\frac{R_2+R_1}{2} + \frac{A_1}{3}\left(R_2^2+R_2R_1+R_1^2\right)\right]}{\sqrt{\left(R_2^2-R_1^2\right) + \frac{4}{3}A_1\left(R_2^3-R_1^3\right) + \frac{A_1}{4}\left(R_2^4-R_1^4\right)}} =$$

$$= \sqrt{\frac{\left[\frac{R_2+R_1}{2} + \frac{A_1}{3}\left(R_2^2+R_2R_1+R_2^2\right)\right]^2}{\left(R_2^2-R_1^2\right) + \frac{4}{3}A_1\left(R_2^3-R_1^3\right) + \frac{A_1}{4}\left(R_2^4-R_1^4\right)}}\sqrt{2gH} =$$

$$= K\sqrt{2gH} \quad (35a)$$

from which,

$$K = \sqrt{\frac{\left[\frac{R_2+R_1}{2} + \frac{A_1}{3}\left(R_2^2+R_2R_1+R_1^2\right)\right]^2}{\left(R_2^2-R_1^2\right) + \frac{4}{3}A_1\left(R_2^3-R_1^3\right) + \frac{A_1}{4}\left(R_2^4-R_1^4\right)}} \quad (36a)$$

When $A_1 = 0$ the last equation evidently reduces to

$$K = \sqrt{\frac{\left(\frac{R_2 + R_1}{2}\right)^2}{R_2{}^2 - R_1^2}} = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}} \quad \text{(See eq. 26a)}$$

as it should.

22. Using the second expression for the F function the equations just mentioned take the following forms

$$\mathbf{p} = \int -\frac{D}{g} R F^2 dR =$$

$$= p_o - \frac{D}{g} C^2 \int R_1^{1+\frac{2}{n}} dR =$$

$$= p_o - \frac{n}{(1+n)} \frac{DC^2}{2g} R^{\frac{2(1+n)}{n}} \qquad (37a)$$

$$V_a = \frac{\int_{R_1}^{R_2} v\,dR}{R_2 - R_1} = C \int_{R_1}^{R_2} R^{1+\frac{1}{n}} dR =$$

$$= \frac{n\,C}{(2n+1)} \frac{R_2^{\frac{2n+1}{n}} - R_1^{\frac{2n+1}{n}})}{(R_2 - R_1)}$$

(38a)

$$p_1 - p_2 = \frac{n}{(1+n)} \frac{DC^2}{2g} \left(R_2^{\frac{2(1+n)}{n}} - R_1^{\frac{2(1+n)}{n}}\right) \qquad (39a)$$

or,

$$V_a = \sqrt{\frac{n(1+n)\left(R_2^{\frac{4n+2}{n}} - R_1^{\frac{4n+2}{n}}\right)}{(2n+1)^2 \left(R_2^{\frac{2+2n}{n}} - R_1^{\frac{2+2n}{n}}\right)(R_2 - R_1)^2}} \sqrt{2gH} =$$

$$= K\sqrt{2gH} \qquad (40a)$$

from which,

$$K = \sqrt{\frac{n(n+1)\left(R_2^{\frac{4n+2}{n}} - R_1^{\frac{4n+2}{n}}\right)}{(2n+1)^2\left(R_2^{\frac{2+2n}{n}} - R_1^{\frac{2+2n}{n}}\right)(R_2-R_1)^2}} =$$

$$= \frac{1}{2}\sqrt{\frac{R_2^{\frac{54}{13}} - R_1^{\frac{54}{13}}}{\left(R_2^{28/13} - R_1^{28/13}\right)(R_2-R_1)^2}} \qquad (41a)$$

for $n = 13$, which is the value found experimentally for water.

COMPRESSIBLE FLUIDS

23. Referring back to the definition of flux for stream line flow, we note, that upon second thought, the logical thing to consider as the flux passing across the unit height cylindrical surface, enveloping any curve AB, is the weight rather than the volume of fluid crossing the cylindrical surface.

24. For when we consider compressible fluids, i.e., gases and vapors, the density of the fluid when it crosses the curve AOB is not necessarily the same as when it crosses the curves A O_1B, AO_2B etc., the density varying from point to point. Also, the same weights of fluid must cross the curves AOB and AO_1B in a given time, or else there will be an increase or decrease of the weight of fluid contained within the space enveloped by AOB_1O A, as

time passes, a condition~~,~~ which for steady flow is impossible. But, if the weight of the fluid crossing AOB *is* the same, while the density may *not* be the same, as that crossing the curves AOB, the *volumes* of fluid crossing curves A O B and A O_1B are *not* necessarily equal.

25. Therefore, in the case of compressible fluids, we shall consider as flux across *AB* the *weight* crossing the unit height cylindrical surface enveloping any curve connecting the points A and C. In the case of *incompressible* fluids, it is immaterial if the *volume* instead of the weight is considered, since the density of a liquid does *not* vary from point to point.

26. If we name the flux of a compressible fluid gravitational or "g" flux, we may state again, as in the case of incompressible fluids, that, for flow along hyperbolic stream lines the "g" flow is some function of the hyperbolas; i.e.,

$$\text{"g" flux} = f'(-xy) = A_1 - B_1xy + C_1x^2y^2 - D_1x^3y^3 + \dots \qquad (42a)$$

(Compare with eq. 8a)

27. Furthermore, in a way similar to that used for deriving equation (2a) to (3"a) inclusive, it can be shown that,

$$\text{"g" flux increase} = \partial f = \rho V_x dy \qquad (43a)$$

$$\text{"g" flux decrease} = \partial f = -\rho V_y \partial x \qquad (44a)$$

where, $\rho = \frac{D}{g}$ is mass per unit volume or "volumetric" density, while D is weight per unit volume or "gravitational" density. The last two equations can be rewritten as follows:

$$\rho V_x = \frac{\partial f}{\partial y} = -B_1 x + 2C_1 x^2 y - 3D_1 x^3 y^2 + \ldots =$$
$$= -x\left(B_1 - 2C_1 xy + \ldots\right) = -x F^1(-xy) \qquad (45a)$$

$$\rho V_y = -\frac{\partial f}{\partial x} = -\left(-B_1 y + 2C_1 xy^2 - 3D_1 x^2 y^3 + \ldots\right) =$$
$$= y\left(B_1 - 2C_1 xy + \ldots\right) = +y F^1(-xy) \qquad (46a)$$

Also, we have,

$$\rho V = \sqrt{(\rho V_x)^2 + (\rho V_y)^2} = F^1(-xy)\sqrt{x^2+y^2} =$$
$$= r F^1(-xy) \qquad (47a)$$

28. The law of centrifugal force as affecting pressure distribution along the line of symmetry, holds for any fluid, whether incompressible or compressible as long as plane hyperbolic stream line flow exists. We may, therefore, still write,

$$\frac{dp}{dR} = -\frac{D}{g}\frac{V^2}{R} \qquad \text{(see eq. 18a)}$$

The only difference is the fact that whereas, for incompressible fluids, the density D at any point along the line of symmetry of the elbow is the

same as the average density D_a in the pipe, for compressible fluids the density D is a variable.

29. It can, however, be shown that, even for compressible fluids, the variation of the density, D, along the line of symmetry is very small, and the value of D at any point along the line of symmetry is very close to that of the average density Da in the pipe.

30. Thus, taking as an illustration the case of dry steam flowing at the rate of 100'/sec at 100 lbs. gauge pressure, we note from steam tables that the average density of the steam is $Da = .257$ lbs/cu. ft. Now, along the line of symmetry, the velocity v according to equation (47a) is proportional to the radius vector, R. ($r = R$ along the line of symmetry). Therefore, at the origin, where $R = 0$, $v = 0$, also, and all the energy is in the form of "pressure" potential energy with no kinetic energy being present. Calling the pressure at the origin p_o, and the average pressure and average velocity in the pipe, p_a and v_a, respectively, we may, in accordance with the law of conservation of energy write,

$$p_o = p_a + \frac{D v_a^2}{2g} \qquad (48a)$$

if the energy is uniformly distributed among all the

stream lines, i.e.,

$$Po - Pa = \frac{D v_a^2}{2g} = \frac{.257 \times 100^2}{2 \times 32.2} = 40 \text{ \#/sq. ft.} =$$

$$= .278 \text{ lbs./sq.in.}$$

is the maximum excess of the pressure at any point along the line of symmetry over the average pressure in the pipe.

31. But, according to the steam tables, the density variation per 1 lbs/ sq. in. change of pressure., around the neighborhood under consideration, is only about 1%. Therefore, the maximum possible density along the line of symmetry must be only

$$1 \times .278 = .278\%$$

or negligibly more than the average density.

32. Similarly, since the usual pressure difference obtained between the outside and inside curves of the hyperbolic elbow are between 1 and 2 inches of mercury or between .492 and .984 lbs/sq.in., the average minimum possible pressure along the line of symmetry must be only

$$.984 - .278 = .706 \text{ lbs/sq.in.}$$

smaller than the average pressure in the pipe. This corresponds to the minimum density being of a

$$1 \times .706 = .706\%$$

smaller value than the average density in the pipe.

32. Even if the energy is not uniformly distributed among all stream lines, e.g., as in the case of flow with friction, the maximum value of $p_o - p_a$ could not exceed .278 lbs/sq. in. much more than say 100%. In short, the maximum density variations along the line of symmetry as compared with the average density in the pipe, may, therefore, be neglected.

33. Therefore, if along the line of symmetry we neglect the variation of density, we may write,

$$V = \frac{r}{\rho} F'(-xy) = r F(-xy) \quad (48a)$$

$$\frac{dp}{dR} = -\frac{D}{g}\frac{v^2}{R} = -\frac{D}{g} R F^2 \quad (49a)$$

where,

$$F(-xy) = \frac{1}{\rho} F'(-xy) \quad \text{(Compare with eq.30a)}$$

34. By analyzing the pressure distribution along the line of symmetry for steam flowing through an hyperbolic elbow, as explained in the paper proper, the expression for F (xy) can be found. The nature of the "F" function for steam as well as the expressions for velocity, v, pressure, p, and flow meter constant k involved in the relation

$$V_a = K\sqrt{2g\,H}$$

are all derived in the paper proper to which the reader is referred for details as well as for final results.

APPENDIX B

General Theory of Plane Hyperbolic Stream Line Flow

1. In Lamb's Hydrodynamics (P.216 - P. 220), the reader will find that, for steady plane flow with friction, Euler's general equations of motion take the following forms:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial \theta}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \qquad (1\ \mathrm{B})$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial \theta}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) \qquad (2\mathrm{B})$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{3}\frac{\mu}{\rho}\frac{\partial \theta}{\partial z} \qquad (3\ \mathrm{B})$$

where

Vx = velocity component along the x-axis

Vy = " " " y-axis

$\theta = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

p = pressure at any point (x,y)

ρ = volumetric density = $\frac{D}{g}$ where

D = gravitational "

μ = co-efficient of viscosity

$\frac{\mu}{\rho}$ = "kinematic" viscosity

2. These so called "dynamic" equations are simply expressing the law of conservation of energy in the form of differential equations, stating that the infinitessimal change of kinetic energy is balanced by an equal and opposite change of pressure potential energy and by friction. These equations are also usually supplemented by the so-called "kinematical"

equations of continuity and of boundary conditions, viz,

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} = 0 \qquad (4B)$$

and

$$V_x \sin\theta + v_y \cos\theta = 0 \qquad (5B)$$

The former equation states that the mass passing through a unit volume at any point does not change with time, since the motion is steady. The second equation states that the total volume of fluid crossing the boundary line is zero. (The direction of the boundary makes an angle θ with the x-axis)

3. For incompressible fluids, the equation of continuity is simplified to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (6B)$$

since the diversity,p, for this case is a constant. Also, since now

$$\theta = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

the terms

$$1/3\,\mu\frac{\partial\theta}{\partial x}, \quad 1/3\,\mu\frac{\partial\theta}{\partial y}, \quad 1/3\,\mu\frac{\partial\theta}{\partial z}$$

drop out from the dynamic equations and equation (3B) becomes,

$$\frac{\partial p}{\partial z} = 0$$

4. For stream line flow, the conception of flux, as explained in Appendix A, gives us the equations.

$$\rho v_x = \frac{\partial f}{\partial y} \qquad \text{(see eq. 45A)}$$

$$\rho v_y = -\frac{\partial f}{\partial x} \qquad \text{(see eq. 46A)}$$

for compressible fluids, and

$$v_x = \frac{\partial f}{\partial y} \qquad \text{(See eq. 2"A)}$$

$$v_y = -\frac{\partial f}{\partial x} \qquad \text{(See eq. 3"A)}$$

for incompressible fluids. The former can by differention be changed to

$$\frac{\partial(\rho v_x)}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \qquad (7\ B)$$

$$\frac{\partial(\rho v_y)}{\partial y} = -\frac{\partial^2 f}{\partial x \partial y} \qquad (8\ B)$$

which when combined give,

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} = 0 \qquad (9\ B)$$

The last equation is evidently the same as the equation of continuity for compressible fluids (see eq. 4 B). In a similar manner, the flux equations for incompressible fluids may be shown to check with their equation of continuity, as given by equation (6 B).

5. In addition, along the boundary line, just as along any of the stream lines,

$$\frac{v_y}{v_x} = -\tan\phi \qquad (10B)$$

where ϕ is the negative angle made by the x-axis with the tangent to the boundary or stream line curve (see Fig. 2 of paper proper). The last equation may be rewritten as,

$$v_y = -v_x \tan\phi = -v_x \frac{\sin\phi}{\cos\phi},$$

or,

$$v_x \sin\phi + v_y \cos\phi = 0$$

which is the same as the boundary equation (5 B).

6. It is, therefore, evident that the flux equations may, for stream line flow be used to replace the kinematical equations of continuity and of boundary conditions.

7. For hyperbolic stream lines, it was shown in Appendix A that,

$$\text{Flux} = F(-xy) = A - B\,xy + cx^2 y^2 - \ldots.$$

(see Eq. 84)

and

$$Vx = -x\,F\,(-xy) \qquad \text{(See eq. 27a)}$$

$$Vy = -y\,F\,(-xy) \qquad \text{(See eq. 28a)}$$

for incompressible fluids. While for compressible fluids,

$$\text{"g" flux} = F\,(xy) = A_1 - B_1\,xy + C_1 x^2 y^2 - \ldots.$$

(see eq. 42_a)

and,

$$\rho v_x = -x\,F'(-xy) \qquad \text{(see eq. } 45_a\text{)}$$

$$pv_y = y\,F'(-xy) \qquad \text{(see eq. } 46_a\text{)}$$

8. Using these expressions for the velocity components in the case of incompressible fluids, the general dynamic equation for the latter can be simplified to

$$x F^2 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[2y F_1 - x F_2 (x^2+y^2)\right] \qquad (11B)$$

$$y F^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[-2x F_1 + y F_2 (x^2+y^2)\right] \qquad (12\ B)$$

as shown in Appendix C, Calculation details #1, if the variation of velocity with Z is considered negligible; i.e. if

$$\frac{\partial v_x}{\partial z} = 0 \quad , \text{and} \quad \frac{\partial v_y}{\partial z} = 0$$

In the last equations, F_1 and F_2 are the first and second derivatives of the F function.

9. For analyzing the pressure distribution along the line of symmetry, we proceed as follows:

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial r} \qquad (13B)$$

Since, $x = r\cos\theta$ and $y = r\sin\theta$,

we get, $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$,

or,

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial x}\cos\theta + \frac{\partial p}{\partial y}\sin\theta = \frac{\partial p}{\partial x}\frac{x}{r} + \frac{\partial p}{\partial y}\frac{y}{r} = $$
$$= \frac{1}{r}\left(x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y}\right) \qquad (13\ B)$$

From equation (11 B and (12 B), we get

$$\frac{\partial p}{\partial x} = -\rho x F^2 + \mu\left(2yF_1 - xr^2F_2\right) \qquad (14\ B)$$

and

$$\frac{\partial p}{\partial y} = -\rho y F^2 - \mu\left(2xF_1 - yr^2F_2\right) \qquad (15\ B)$$

Combining the last two equations with eq. (13B) we get,

$$\frac{\partial p}{\partial r} = \frac{1}{r}\left[-\rho x^2F^2 + \mu x\left(2yF_1 - xr^2F_2\right)\right] +$$

$$+\frac{1}{r}\left[-\rho y^2F^2 - \mu y\left(2xF_1 - yr^2F_2\right)\right] =$$

$$= \frac{1}{r}\left[-\rho F^2\left(x^2+y^2\right) - \mu F_2 r^2\left(x^2-y^2\right)\right] =$$

$$= \frac{1}{r}\left[-\rho r^2F^2 - \mu r^2F_2\left(r^2\cos^2\theta - r^2\sin^2\theta\right)\right] =$$

$$= -\rho rF^2 - \mu r^3F_2 \cos 2\theta =$$

$$= -\frac{D}{g} rF^2 - \mu r^3F_2 \cos 2\theta$$

(16B)

10. Along the line of symmetry, $\theta = 45^\circ$ and $\cos 2\theta = \text{Cos } 90 = 0$; also $r = R$. Therefore, the equation for pressure distribution along the line of symmetry takes the comparatively simple form,

$$\frac{dp}{dR} = -\frac{D}{g} R F^2$$

(17B)

which is the same as that derived in Appendix A on the basis of analyzing the effects of [redacted]

the centrifugal forces, along the line of symmetry.

11. It may be remarked that, for the sake of completeness, since the expression for $\frac{\partial p}{\partial r}$ (see eq. 16 B) was derived, on the basis of the angle θ remaining constant, another expression for $\frac{\partial p}{\partial \theta}$ ought to be derived on the basis of the radius vector, r, remaining constant. Thus,

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \theta} \qquad (18'B)$$

The last equation can be shown to take the final form,

$$\frac{\partial p}{\partial \theta} = - 2\mu r^2 \left(F_1 - \frac{r^2}{2} F_2 \sin 2\theta\right) \qquad (18B)$$

(See Appendix C Calculation Details #2)

12. By differentiating the expression for $\frac{\partial p}{\partial r}$ (eq. 16B) with respect to $\underline{\theta}$ and the expression for $\frac{\partial p}{\partial \theta}$ (eq. 18'B) with respect to $\underline{R}$, we get as shown in appendix C, Calculation Details #3,

$$\frac{\partial^2 p}{\partial r \partial \theta} = 2(x^2-y^2)\rho r F F_1 + \left[2F_2 \sin 2\theta + (x^2-y^2) F_3 \cos 2\theta\right]\mu r^3 \qquad (19B)$$

and

$$\frac{\partial^2 p}{\partial \theta \partial r} = 4\mu r F_1 - 6\mu r^3 F_2 \sin 2\theta - \mu r^5 F_3 \sin^2 2\theta \qquad (20B)$$

from which

$$2 \rho r (x^2-y^2) F F_1 + \mu r^3\left[2F_2 \sin 2\theta + (x^2-y^2) F_3 \cos 2\theta\right] =$$

$$= 4\mu r F_1 - 6\mu r^3 F_2 \sin 2\theta + \mu r^5 F_3 \sin^2 2\theta \qquad (21B)$$

13. Since F, F_1, F_2 are all functions of -xy, we must eliminate , r, and $\underline{\theta}$ from the last equation in order to find the relation between these functions and (x,y). First, however, we divide the equation through by and collect similar functions together and we get,

$$2p(x^2-y^2)FF_1 - 4\mu F_1 + 8\mu r^3 F_2 \sin \underline{2\theta} + \\ +F_3\mu r^3\left[(x^2-y^2)\cos 2\theta - r^2\sin^2 2\theta\right] = 0 \qquad (22B)$$

When $\underline{r}$ and $\underline{\theta}$ are eliminated from the last equation, as shown in Appendix C, calculation Details #4, we get,

$$2p(x^2-y^2)FF_1 - 4\mu F_1 + 16\mu xy\sqrt{x^2+y^2}\quad F_2 + \\ +\mu\sqrt{(x^2+y^2)}\;(x^2-y^2)F_3 - 4\mu\sqrt{x^2+y^2}\,x^2y^2F_3 = 0 \qquad (23B)$$

14. If F, F_1, F_2 and F_3 are functions of -xy, the last differential equation ought to contain only these functions their derivates and (-xy) terms without any other $\underline{X}$ and $\underline{Y}$ expressions. The fact that this equation does in addition contain the terms (x^2+y^2) and (x^2-y^2) indicates that, theoretically for hyperbolic stream lines extending for $x = +\infty$ $y = 0$ to $x = 0$ $y = +\infty$ plane stream line motion cannot exist throughout the full lengths of the stream lines on account of the cumulative disturbance of friction. Practically, however, due to the comparatively short fractions of the total hyperbolic stream lines included in the hyperbolic elbow, the cumulative effect

of friction is too small to serously disturb the plane stream line flow, as evidenced by the small difference between the actual water flow meter co-efficients and the theoretical co-efficients for frictionless flow.

15. Attacking the same question from another point of view, we find that

$$\frac{\partial^2 p}{\partial x \partial y} = 2\rho x^2 F F_1 + \mu(2F_1 - 4xyF_2 + x^4F_3 + x^2y^2F_3) \text{ and}$$

$$\frac{\partial^2 p}{\partial y \partial x} = 2\rho y^2 F F_1 - \mu(2F_1 - 4xyF_2 + y^4F_3 + x^2y^2F_3) \quad (25B)$$

$$\text{or, } \underline{2\rho F F_1 (x^2-}y^2) + \mu\left[4F_1 - 8xyF_2 + (x^4+y^4)F_3 + 2x^2y^2F_3\right] = 0$$

(See Appendix C Calcul Details #5)

16. As an approximation we proceed as follows:

For $x = -y$ or along the line of symmetry, the last equation becomes

$$4F_1 + 8x^2F_2 + 4x^4F_3 = 0 \qquad (26B)$$

Let $x^2 = -xy = z$;

Then,

$$4F_1(-xy) + 8x^2F_2(-xy) + 4x^4F_3(-xy) =$$

$$= 4F_1(Z) + 8zF_2(z) + 4z^2F_3(z) =$$

$$= 4\frac{dF}{dz} + 8z\frac{d^2F}{dz^2} + 4z^2\frac{d^3F}{dz^3} = 0$$

or

$$z^2\frac{d^3F}{dz^3} + 2z\frac{d^2F}{dz^2} + \frac{dF}{dz} = 0 \qquad (27B)$$

17. As shown in Appendix C, Calcul. details #6, the solution of the last differential equation is,

$$F(z) = C_1 + B_1 z^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\log z\right) + B_2 z^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\log z\right) \qquad (28B)$$

Therefore, we have,

$$F(x^2) = C_1 + B_1\, x \cos\left(\sqrt{3}\log x\right) + B_2 x \sin\left(\sqrt{3}\log x\right) \qquad (29B)$$

18. If density variation were neglected, then the equations for compressible fluids would become the same as those for incompressible fluids. We would have then the same expression for F as that given in the last equation. But the experimentally found value of F in the case of steam is,

$F\left(\frac{R^2}{2}\right) = \frac{B}{\sqrt{R}}\sqrt{1-.775\cos(28.5R+2.32)}$ (see eq.73) and

therefore, the experimental results satisfy the equation,

$$F(x^2) = \frac{B}{\sqrt[4]{2}\sqrt{x}}\sqrt{1-.775\cos(28.5\sqrt{2}x+2.32)} \qquad (30B)$$

19. An inspection of and comparison be-

tween equations (29B) and (30B) shows that both of them involve a periodic function. Furthermore, the "F" curve for water can also be made to satisfy equation (29B), if this "F" curve is considered to be a periodic curve of a much larger amplitude than the one for steam.

20. The question then arises: Is equation the general rational expression for the "F" function? If so, what properties of the fluid do the constants B_1, and B_2 depend upon? These are some of the important points left yet for future investigation.

21. Then again, the question remains, as to what modifications, if any, must be made of eq. (16B) for $\frac{dp}{dr}$, so as to make it applicable for calculating pressure differences between any two points on the outside and on the inside curves of a hyperbolic elbow, and not just for the line of symmetry. Thus let us suppose for instance, that the equation

$$\frac{dp}{dr} = -\frac{D}{g} r F^2 - \mu r^3 F_2 \cos 2\theta \quad \text{(see eq.16B)}$$

is correct in a general sense for any values of θ, or that the integral of this expression does not involve any additional function of θ as a constant of integration. Then, for the case of

$$F(R) = C R^{\frac{1}{n}} \quad \begin{array}{l}\text{(see eq. 37)}\\ \text{(or eq. 31A)}\end{array}$$

we have $F\left(\frac{R^2}{2}\right) = 2^{\frac{1}{2n}} C \left(\frac{R^2}{2}\right)^{\frac{1}{2n}}$ (31B)

and

$$F(-xy) = 2^{\frac{1}{2n}} C (-xy)^{\frac{1}{2n}} \tag{32B}$$

The integration for the expression for $\frac{dp}{dr}$ then gives the following general expression for the pressure, p, at any point:

$$p = p_0 - \frac{D}{g}\int r F^2 dr - \mu \cos 2\theta \int r^3 F_2 dr \tag{33B}$$

22. Now if we take two points, say on the outside curve of the hyperbolic elbow, symmetrically situated with respect to the line of symmetry, their radii vectors, r, will be the same while their angles with the x-axis will necessarily be θ and $90-\theta$. Then the difference of pressure between these two points will be, according to equation (33B)

$$p_{\theta} - p_{90-\theta} = \mu\left[-\cos(2\theta) + \cos 2(90-\theta)\right]\int r^3 F_2 dR =$$

$$= \mu\left[-\cos(2\theta) + \cos(180-2\theta)\right]\int r^3 F_2 dr =$$

$$= -2\mu\cos 2\theta \int r^3 F_2 dr =$$

$$= -2\mu\cos 2\theta \, \psi(r) \tag{34B}$$

In other words, the difference of pressure between two such symmetrical points involves the co-efficient of viscosity, μ, of the fluid considered and some definite function of the expression for $F(-xy)$, If the latter is known because of experimental investigation of the pressure distribution along the line of

symmetry as explained in the paper proper, the value of μ may be found by analyzing the pressure distribution along the external and internal curves of the elbow and then using equation (34B) just derived. Thus, as shown in Appendix C, Calculation Details #7, for the case of

$$F(R) = C R^{\frac{1}{n}}$$

we get,

$$P90-\theta - p_\theta = -2\mu C \frac{(2n-1) m^{\frac{2n-1}{n}} \sqrt{m^4-1}}{n} r^{\frac{1}{n}} \quad (35B)$$

$(m = \frac{r}{R}$ where $R =$ shortest radius vector)

In cases like steam or other fluids whose coefficients of viscosity are not known, the above suggests an experimental method of finding μ for different pressures, temperatures, qualities, etc. Furthermore, when μ is known, e.g. as in the case of water, equation (34B) suggests an experimental method of finding $\psi(r)$ and therefore, also $F(r)$, as a check on the line of symmetry method, or the results may disclose the actual law of general pressure distribution which may modify the general expression for $\frac{dp}{dr}$ and F and thus correct for any possible discrepancy caused by the disturbing effect of friction. The investigation outlined is the next step the author expects to take in the investigation of the properties of the hyperbolic elbow.

22. Finally, higher velocities seem to slightly increase the coefficient $\underline{k}$ of the relation,

$$V = K\sqrt{2gH}$$

in the case of water, and to decrease this coefficient, in the case of steam, suggesting that, *possibly* the relation between average velocity and pressure difference is of a form somewhat different than that given above; e.g.,

$$V = C_1 \sqrt{2gH} \pm C_2$$

so that:

$$K = \frac{V}{\sqrt{2gH}} = \frac{C_1\sqrt{2gH} \pm C_2}{\sqrt{2gH}} =$$

$$= C_1 \pm \frac{C_2}{\sqrt{2gH}} \qquad (30B)$$

showing that, for the case of $+ C_2$, as the velocity, $\underline{V}$, increases, the pressure difference head, $\underline{H}$ increases, and $\frac{C}{\sqrt{2gH}}$ *decreases* thereby *decreasing K*. Or again, it may be that at higher velocities the straightening and stream line effect is not as complete as in the case of lower velocities, still in the case of steam, not only just as good, but even *better* results were obtained for higher velocities than for lower velocities. This unexplained effect of velocity upon the flow coe-fficient is another point to be looked into.

23. In short, even though, from a purely practical point of view, the final aim of a definite flow meter coefficient for a hyperbolic elbow has been successfully reached, from a scientific point of view, it would be very desirable to supplement the data so far secured by additional data of a character which would make the theorectical analysis of the hyperbolic elbow complete.

APPENDIX C

MATHEMATICAL DETAILS

$$V_x = -x\,F(-xy) = -x\left(B - 2Cxy + 3Dx^2y^2 - \dots\right)$$

$$V_y = y\,F(-xy) = y\left(B - 2Cxy + 3Dx^2y^2 - \dots\right)$$

CALCULATION DETAILS #1

$$\frac{\partial V_x}{\partial x} = -F - x\frac{\partial F}{\partial x} \quad ; \quad \frac{\partial v_x}{\partial y} = -x\frac{\partial F}{\partial y}$$

$$\frac{\partial V_y}{\partial x} = y\frac{\partial F}{\partial x} \quad ; \quad \frac{\partial v_y}{\partial y} = F + y\frac{\partial F}{\partial y}$$

$$\frac{\partial F(-xy)}{\partial x} = F_1(-xy)\frac{\partial(-xy)}{\partial x} = -yF_1$$

$$\frac{\partial F(-xy)}{\partial y} = F_1(-xy)\frac{\partial(-xy)}{\partial y} = -xF_1$$

$$(1)\quad V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} = -xF\left\{-F - x\left[-yF_1(-xy)\right]\right\} + {}$$
$$+yF\left\{-x\left[-xF_1(-xy)\right]\right\} =$$

$$= xF^2 - x^2yFF_1 + yx^2FF_1 = xF^2$$

$$(2)\quad V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} = -xF\left\{y\left[-yF_1\right]\right\} + yF\left\{F + y\left[-xF_1\right]\right\} =$$

$$= +xy^2FF_1 + yF^2 - y^2xFF_1 = +yF^2$$

$$\frac{\partial V_x}{\partial x} = -F - x\frac{\partial F}{\partial x} = -F - x\left(-yF_1\right) = -F + xyF_1$$

$$\frac{\partial V_x}{\partial y} = -x\frac{\partial F}{\partial y} = -x\left(-xF_1\right) = x^2F_1$$

$$\frac{\partial V_y}{\partial x} = y\frac{\partial F}{\partial x} = y\left(-yF_1\right) = -y^2F_1$$

$$\frac{\partial V_y}{\partial y} = F + y\frac{\partial F}{\partial y} = F + y\left(-xF_1\right) = F - xyF_1$$

$$\frac{\partial F1}{\partial x}(-xy) = F_2\frac{\partial(-xy)}{\partial x} = -yF_2$$

$$\frac{\partial F_1(-xy)}{\partial y} = F_2\frac{\partial(-xy)}{\partial y} = -xF_2$$

$$\frac{\partial^2 V_x}{\partial x2} = \frac{\partial\left(\frac{\partial v_x}{\partial x}\right)}{\partial x} = -\frac{\partial F}{\partial x} + yF_1 + xy\frac{\partial F_1}{\partial x} =$$

$$= +yF1 + yF_1 - xy^2F_2 = 2yF_1 - xy^2F_2$$

$$\frac{\partial^2 V_x}{\partial Y2} = \frac{\partial\left(\frac{\partial v_x}{\partial y}\right)}{\partial y} = x^2\frac{\partial F_1}{\partial y} = -x^3F_2$$

$$\frac{\partial^2 V_y}{\partial x^2} = \frac{\partial\left(\frac{\partial v_y}{\partial x}\right)}{\partial x} = -y^2\frac{\partial F_1}{\partial x} = +y^3F_2$$

$$\frac{\partial^2 V_y}{\partial y^2} = \frac{\partial\left(\frac{\partial V_y}{\partial y}\right)}{\partial y} = \frac{\partial F}{\partial y} - xF_1 - xy\,\frac{\partial F_1}{\partial y} =$$

$$= -xF_1 - xF_1 + x^2yF_2 =$$

$$= -2xF_1 + x^2yF_2$$

$$(3) \quad \frac{\mu}{\rho}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) = \frac{\mu}{\rho}\left(2yF_1 - xy^2F_2 - x^3F_2\right) =$$

$$= \frac{\mu}{\rho}\left[2yF_1 - xF_2\left(x^2+y^2\right)\right]$$

$$(4) \quad \frac{\mu}{\rho}\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right) = \frac{\mu}{\rho}\left(x^2yF_2 - 2xF_1 + y^3F_2\right) =$$

$$= \frac{\mu}{\rho}\left[-2xF_1 + yF_2\left(x^2+y^2\right)\right]$$

CALCULATION DETAILS #2

$x = r\cos\theta$, $Y = r\sin\theta$

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta; \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial p}{\partial \theta} = -r\sin\theta\,\frac{\partial p}{\partial x} + r\cos\theta\,\frac{\partial p}{\partial y} = -y\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y}$$

$$x\,F^2 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(2yF_1 - xr^2F_2\right)$$

$$yF^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(-2xF_1 + yr^2F_2\right)$$

$$\frac{\partial p}{\partial x} = -\rho xF^2 + \mu\left(2yF_1 - xr^2F_2\right)$$

$$\frac{\partial p}{\partial y} = -\rho yF^2 - \mu\left(2xF_1 - yr^2F_2\right)$$

(1) $$-y\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} = \rho xyF^2 - \mu\left(2y^2F_1 - xyr^2F_2\right) -$$

$$-\rho xyF^2 - \mu\left(2x^2F_1 - xyr^2F_2\right) =$$

$$= -\mu\left(2r^2F_1 - 2xyr^2F_2\right) =$$

$$= -2\mu r^2\left(F_1 - xyF_2\right)$$

$$\frac{\partial p}{\partial \theta} = -y\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial y} = -2\mu r^2\left(F_1 - xyF_2\right) =$$

$$= 2r^2\mu\left(F_1 - r^2\cos\theta\sin\theta F_2\right) =$$

$$= 2r^2\mu\left(F_1 - \frac{r^2}{2}F_2\sin 2\theta\right)$$

CALCULATIONS DETAILS #3

$$\frac{\partial p}{\partial r} = -\rho rF^2 - \mu r^3F_2\cos 2\theta$$

(1d) $$\frac{\partial^2 p}{\partial r\partial \theta} = -2\rho rF\frac{\partial F}{\partial \theta} - \mu r^3\left(-2F_2\sin 2\theta + \frac{\partial F_2}{\partial \theta}\cos 2\theta\right)$$

$$\frac{\partial F(-xy)}{\partial \theta} = F_1(-xy)\,\frac{\partial(-xy)}{\partial \theta}$$

$$\frac{\partial(-xy)}{\partial \theta} = \frac{\partial(-xy)}{\partial x}\,\frac{\partial x}{\partial \theta} + \frac{\partial(-xy)}{\partial y}\,\frac{\partial y}{\partial \theta}$$

$$x = r\cos\theta\,; \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y \quad ; \quad \frac{\partial y}{\partial \theta} = r\cos\theta = x$$

$$\frac{\partial(-xy)}{\partial \theta} = (-y)(-y) + (-x)(x) = y^2 - x^2$$

$$\frac{\partial F(-xy)}{\partial \theta} = F_1(-xy)(y^2-x^2) = (y^2-x^2)F_1$$

$$\frac{\partial F_2(-xy)}{\partial \theta} = F_3(-xy)\,\frac{\partial(-xy)}{\partial \theta} = (y^2-x^2)F_3$$

$$(1) \quad \frac{\partial^2 p}{\partial r\partial\theta} = 2pr(x^2-y^2)FF_1 +$$

$$+ \mu r^3\left[2F_2\sin 2\theta + (x^2-y^2)F_3\cos 2\theta\right]$$

$$\frac{\partial p}{\partial \theta} = 2\mu r^2\left(F_1 - \frac{r^2}{2}F_2\sin 2\theta\right)$$

$$(2a) \quad \frac{\partial^2 p}{\partial\theta\partial r} = 2\mu\left(2rF_1 + r^2\frac{\partial F_1}{\partial r}\right) -$$

$$-2\mu r \sin 2\theta \left(2r^3F_2 + \frac{r^4}{2}\frac{\partial F_2}{\partial r}\right)$$

$$\frac{\partial F_1(-xy)}{\partial r} = F_2(-xy)\frac{\partial(-xy)}{\partial r}$$

$$\frac{\partial(-xy)}{\partial r} = \frac{\partial(-xy)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial(-xy)}{\partial y}\frac{\partial y}{\partial r} =$$

$$= -y\frac{\partial x}{\partial r} - x\frac{\partial y}{\partial r}$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial(-xy)}{\partial r} = -y\cos\theta - x\sin\theta =$$

$$= -r\sin\theta\cos\theta - r\cos\theta\sin\theta =$$

$$= -2r\sin\theta\cos\theta = -r\sin 2\theta$$

$$\frac{\partial F_1(-xy)}{\partial r} = -rF_2\sin 2\theta$$

$$\frac{\partial F_2(-xy)}{\partial r} = F_3(-xy)\frac{\partial(-xy)}{\partial r} = -rF_3\sin 2\theta$$

$$(2)\quad \frac{\partial^2 p}{\partial\theta\,\partial r} = 4\mu rF_1 - 2\mu r^3F_2\sin 2\theta -$$

$$- 4\mu r^3 F_2 \sin 2\theta + \mu r^5 F_3 \sin^2 2\theta =$$

$$= 4\mu r F_1 - 6\mu r^3 F_2 \sin 2\theta + \mu r^5 F_3 \sin^2 2\theta$$

CALCULATION DETAILS #4

(1) $8 \mu r^3 F_2 \sin 2\theta = 8\mu r^3 F_2 \, 2 \sin\theta \cos\theta =$

$$16 \mu r x y F_2 = 16 \mu x y \sqrt{x^2+y^2} F_2$$

(2) $F_3 \mu r^3 (x^2-y^2) \cos 2\theta = F_3 \mu r^3 (x^2-y^2)(\cos^2\theta - \sin^2\theta) =$

$$= F_3 \mu r (x^2-y^2)(x^2-y^2) = F_3 \mu \sqrt{x^2+y^2} (x^2-y^2)^2$$

(3) $- F_3 \mu r^5 \sin^2 2\theta = - F_3 \mu r (2r^2 \sin\theta \cos\theta)^2 =$

$$= -4F_3 \mu r x^2 y^2 = -4 F_3 \mu \sqrt{x^2+y^2} x^2 y^2$$

CALCULATION DETAILS #5

$$\frac{\partial p}{\partial x} = - p x F^2 + \mu (2y F_1 - x r^2 F_2)$$

$$\frac{\partial p}{\partial y} = - \rho y F^2 - \mu (2x F_1 - y r^2 F_2)$$

$$\frac{\partial^2 p}{\partial x \partial y} = - 2\rho x F F_1 \frac{\partial(-xy)}{\partial y} + \mu \Big[2F_1 + 2y F_2 \frac{\partial(-xy)}{\partial y} -$$

$$- x^3 F_3 \frac{\partial(-xy)}{\partial y} - 2xy F_2 - x y^2 F_3 \frac{\partial(-xy)}{\partial y} \Big] =$$

$$= 2\rho x^2 F F_1 + \mu\left(2F_1 - 4xyF_2 + x^4F_3 + x^2y^2F_3\right)$$

$$\frac{\partial^2 p}{\partial y \partial x} = -2\rho y F F_1 \frac{\partial(-xy)}{\partial x} - \mu\Big[2F_1 + 2xF_2 \frac{\partial(-xy)}{\partial x} -$$

$$- y^3F_3 \frac{\partial(-xy)}{\partial x} - 2yxF_2 - yx^2F_3 \frac{\partial(-xy)}{\partial x}\Big] =$$

$$= +2\rho y^2 F F_1 - \mu\left(2F_1 - 4xyF_2 + y^4F_3 + x^2y^2F_3\right)$$

$$= +2\rho x^2 FF_1 + \mu\left(2F_1 - 4xyF_2 + x^4F_3 + x^2y^2F_3\right) =$$

$$= 2\rho Y^2 FF_1 - \mu\left(2F_1 - 4xyF_2 + y^4F_3 + x^2y^2F_3\right)$$

or, $2\rho FF_1(x^2-y^2) + \mu\left[4F_1 - 8xyF_2 + (x^4+y^4)F_3 + 2x^2y^2F_3\right]$

$$= 2\rho FF_1(x^2-y^2) + \mu\left[4F_1 - 8xyF_2 + (x^2+y^2)^2F_3\right] = 0$$

CALCULATION DETAILS #6

(1) $$z^2 \frac{d^3F}{dz^3} + 2z\frac{d^2F}{dz^2} + \frac{dF}{dz} = 0$$

Let $z = e^u$ or $u = \log z$; $\frac{\partial u}{\partial z} = \frac{1}{z}$.

Then,

$$\frac{dF}{dz} = \frac{dF}{du}\frac{dU}{dz} = \frac{1}{z}\frac{dF}{du}$$

$$\frac{d^2F}{dz^2} = \frac{d}{dz}\left(\frac{1}{z}\frac{dF}{du}\right) = -\frac{1}{z^2}\frac{dF}{du} + \frac{1}{z}\frac{d^2F}{du^2}\frac{du}{dz} =$$

$$\left(\frac{1}{z}\right)^2 \left(\frac{d^2F}{d\,u^2} - \frac{dF}{dU}\right)$$

$$\frac{d^3F}{d\,z^3} = \frac{d}{d\,z}\left(\frac{1}{z^2}\,\frac{d^2F}{dU^2} - \frac{1}{z^2}\,\frac{dF}{dU}\right) =$$

$$= \frac{-2}{z^3}\,\frac{d^2F}{d\,u^2} + \frac{1}{z^2}\,\frac{d^3F}{dU^3}\,\frac{dU}{dz} + \frac{2}{z^3}\,\frac{dF}{dU} -$$

$$- \frac{1}{z^2}\,\frac{d^2F}{d\,u^2}\,\frac{dU}{dz} = \frac{1}{z^3}\left(\frac{d^3F}{dU^3} - 3\,\frac{d^2F}{dU^2} + 2\,\frac{dF}{dU}\right)$$

Substituting these expressions into (1), we get

$$\frac{1}{Z}\left(\frac{d^3F}{d\,u^3} - 3\,\frac{d^2F}{dU^2} + 2\,\frac{dF}{dU}\right) + \frac{2}{Z}\left(\frac{d^2F}{dU^2} - \frac{dF}{dU}\right) + \frac{1}{Z}\,\frac{dF}{dZ} = 0$$

or

$$(2) \qquad \frac{d^3F}{d\,u^3} - \frac{d^2F}{dU^2} + \frac{dF}{dU} = 0$$

The last equation is a linear differential equation with constant co-efficients and with the right hand member being zero. Its auxiliary equation is,

$m^3 - m^2 + m = 0$, or , $m\,(m^2 \mp m + 1) = 0$,

whose roots are,

$m_1 = 0; m_2 = \frac{1 + \sqrt{1-4}}{2} = \frac{1 + \sqrt{3}\,j}{2}$; $m_3 = \frac{1 - \sqrt{3}\,j}{2}$

Therefore the solution is,

$F = c_1 e^{m_1 U} + c_2 e^{m_2 U} + c_3 e^{m_3 U} =$

$$= c_1 e^0 + c_2 e^{\frac{1+\sqrt{3}}{2} j U} + c_3 e^{\frac{1-\sqrt{3}}{2} j U} =$$

$$= c_1 + c_2 e^{\frac{U}{2}} e^{\frac{\sqrt{3}}{2} Uj} + c_3 e^{\frac{U}{2}} e^{-\frac{\sqrt{3}}{2} Uj} =$$

$$= c_1 + e^{u/2} \left[B_1 \cos\left(\tfrac{\sqrt{3}}{2} U\right) + B_2 \sin\left(\tfrac{\sqrt{3}}{2} U\right) \right]$$

Since, $U = \log z$, $e^U = z$, $e^{\frac{U}{2}} = z^{\frac{1}{2}}$, or

$$(3) \quad F = C_1 + z^{\frac{1}{2}} \left[B_1 \cos\left(\tfrac{\sqrt{3}}{2} \log z\right) + B_2 \sin\left(\tfrac{\sqrt{3}}{2} \log z\right) \right]$$

CALCULATION DETAILS #7

$$F(-xy) = 2^{\frac{1}{2n}} C (-xy)^{\frac{1}{2n}}$$

$$F_1(-xy) = \frac{2^{\frac{1}{2n}}}{2n} C (-xy)^{\frac{1}{2n}-1} = \frac{2^{\frac{1}{2n}-1}}{n} C (-xy)^{\frac{1}{2n}-1}$$

$$F_2(-xy) = \frac{(1-2n)2^{\frac{1}{2n}-1}}{2n^2} C (-xy)^{\frac{1}{2n}-2} =$$

$$= \frac{(1-2n)}{n^2} 2^{\frac{1}{2n}-2} C (-xy)^{\frac{1}{2n}-2} =$$

$$= \frac{(1-2n)}{n^2} 2^{\frac{1}{2n}-2} C \left(\frac{R^2}{2}\right)^{\frac{1}{2n}-2} =$$

$$= \frac{(1-2n)}{n^2} C R^{\frac{1}{n}-4}$$

$$-xy = \frac{R^2}{2} = -r\cos\theta\, r\sin\theta = -\frac{r^2}{2}\sin 2\theta$$

$$R = \sqrt{-r^2\sin 2\theta} = r\sqrt{\sin(-2\theta)} = \frac{r}{m}\ ; \ \sqrt{\sin(-2\theta)} = \frac{1}{m}$$

$$\psi(r) = \int r^3 F_2\, dr = \frac{1-2n}{n^2} C \int r^3 R^{\frac{1}{n}-4}\, dR =$$

$$= \frac{1-2n}{n^2} C \int r^3\, r^{\frac{1}{n}-4} \sin^{\frac{1}{2n}-2}(-2\theta)\, dr = \frac{1-2n}{n^2} C \sin^{\frac{1}{2n}-2}(-2\theta) \int r^{\frac{1}{n}-1}\, dr =$$

$$= \frac{(1-2n)}{n^2} C \sin^{\frac{1}{2n}-2}(-2\theta)\, n r^{\frac{1}{n}}$$

$$p_\theta - p_{90-\theta} = -2\mu \cos 2\theta\, \psi(r) =$$

$$= \frac{-2\mu(1-2n)}{n^2} C \cos 2\theta \sin^{\frac{1}{2n}-2}(-2\theta)\, n r^{\frac{1}{n}}$$

$$\sin(-2\theta = \frac{1}{m^2};\ \cos(-2\theta) = \sqrt{1-\frac{1}{m^4}} = \frac{\sqrt{m^4-1}}{m^2} = \cos 2\theta$$

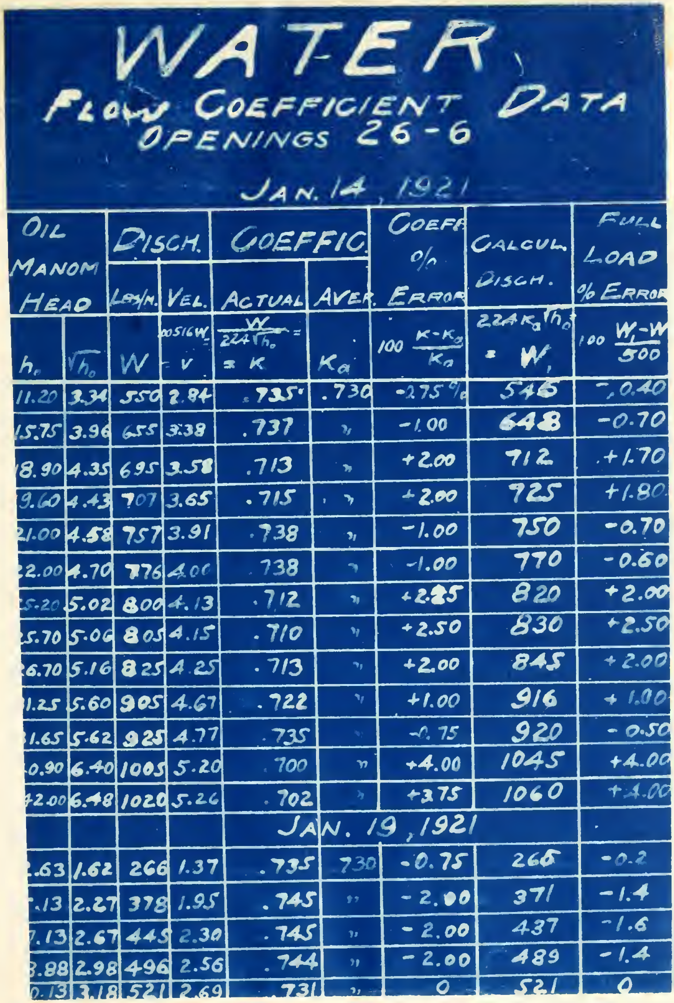
$$p_\theta - p_{90-\theta} = -\frac{2\mu(1-2n)}{n^2} C \frac{\sqrt{m^4-1}}{m^2} m^{4-\frac{1}{n}} n r^{\frac{1}{n}} =$$

$$= \frac{2\mu C(2n-1)\sqrt{m^4-1}\ m^{2-\frac{1}{n}}}{n} r^{\frac{1}{n}}$$

APPENDIX "D"

DATA

TABLE 18



WATER
FLOW COEFFICIENT DATA
OPENINGS 26-6

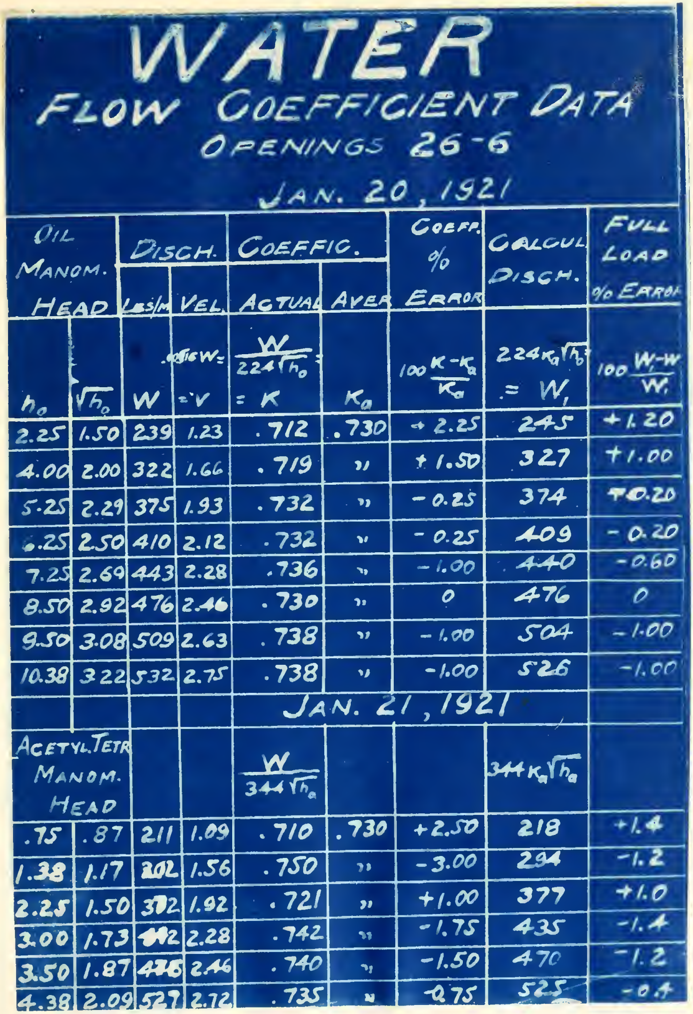
JAN. 14, 1921

OIL MANOM HEAD		DISCH. LBS/M.	DISCH. VEL.	COEFFIC. ACTUAL	COEFFIC. AVER.	COEFF % ERROR	CALCUL. DISCH.	FULL LOAD % ERROR
h_0	$\sqrt{h_0}$	W	$.00516W = v$	$\frac{W}{224\sqrt{h_0}} = K$	K_a	$100\frac{K-K_a}{K_a}$	$224K_a\sqrt{h_0} = W_1$	$100\frac{W_1-W}{500}$
11.20	3.34	550	2.84	.735	.730	-2.75 %	546	-0.40
15.75	3.96	655	3.38	.737	"	-1.00	648	-0.70
8.90	4.35	695	3.58	.713	"	+2.00	712	+1.70
9.60	4.43	707	3.65	.715	"	+2.00	725	+1.80
21.00	4.58	757	3.91	.738	"	-1.00	750	-0.70
2.00	4.70	776	4.00	.738	"	-1.00	770	-0.60
5.20	5.02	800	4.13	.712	"	+2.25	820	+2.00
5.70	5.06	805	4.15	.710	"	+2.50	830	+2.50
6.70	5.16	825	4.25	.713	"	+2.00	845	+2.00
1.25	5.60	905	4.67	.722	"	+1.00	916	+1.00
1.65	5.62	925	4.77	.735	"	-0.75	920	-0.50
0.90	6.40	1005	5.20	.700	"	+4.00	1045	+4.00
42.00	6.48	1020	5.26	.702	"	+3.75	1060	+4.00
				JAN. 19, 1921				
.63	1.62	266	1.37	.735	730	-0.75	265	-0.2
.13	2.27	378	1.95	.745	"	-2.00	371	-1.4
.13	2.67	445	2.30	.745	"	-2.00	437	-1.6
3.88	2.98	496	2.56	.744	"	-2.00	489	-1.4
0.13	3.18	521	2.69	.731	"	0	521	0

APPENDIX "D"

D A T A

T A B L E 19



WATER

FLOW COEFFICIENT DATA

OPENINGS 26-6

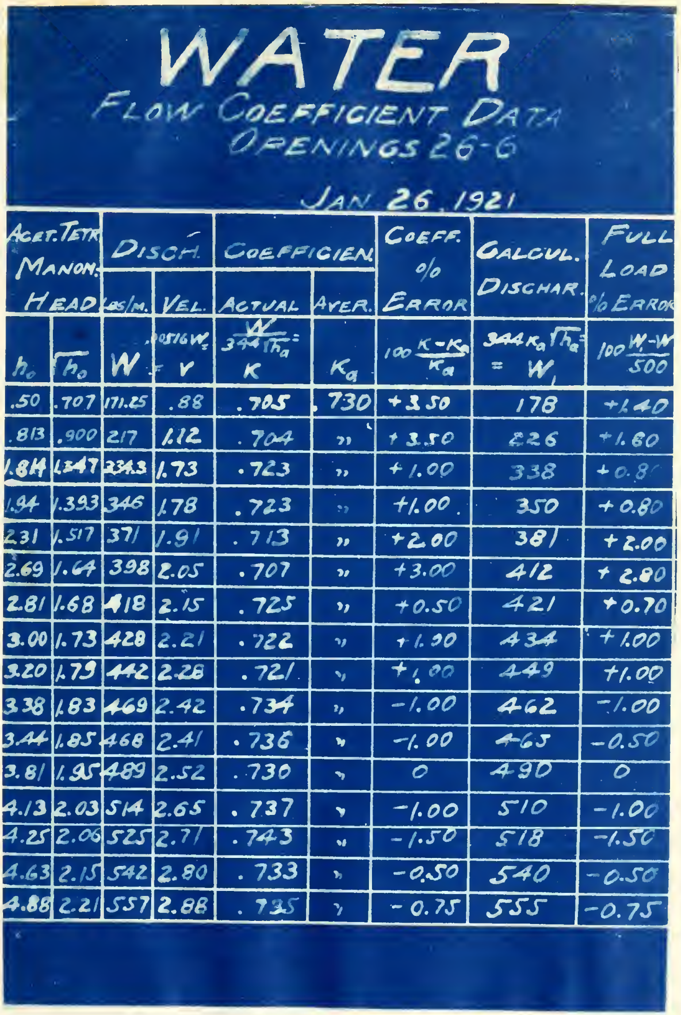
JAN. 20, 1921

OIL MANOM. HEAD		DISCH. LBS/M	VEL.	COEFFIC. ACTUAL	AVER	COEFF. % ERROR	CALCUL DISCH.	FULL LOAD % ERROR
h_o	$\sqrt{h_o}$	W	$.00516W = V$	$\frac{W}{224\sqrt{h_o}} = K$	K_a	$100\frac{K-K_a}{K_a}$	$224K_a\sqrt{h_o} = W_1$	$100\frac{W_1-W}{W_1}$
2.25	1.50	239	1.23	.712	.730	+ 2.25	245	+1.20
4.00	2.00	322	1.66	.719	"	+ 1.50	327	+1.00
5.25	2.29	375	1.93	.732	"	- 0.25	374	+0.20
6.25	2.50	410	2.12	.732	"	- 0.25	409	- 0.20
7.25	2.69	443	2.28	.736	"	- 1.00	440	- 0.60
8.50	2.92	476	2.46	.730	"	0	476	0
9.50	3.08	509	2.63	.738	"	- 1.00	504	- 1.00
10.38	3.22	532	2.75	.738	"	- 1.00	526	- 1.00
				JAN. 21, 1921				
ACETYL. TETR. MANOM. HEAD				$\frac{W}{344\sqrt{h_a}}$			$344K_a\sqrt{h_a}$	
.75	.87	211	1.09	.710	.730	+2.50	218	+1.4
1.38	1.17	302	1.56	.750	"	-3.00	294	-1.2
2.25	1.50	372	1.92	.721	"	+1.00	377	+1.0
3.00	1.73	442	2.28	.742	"	-1.75	435	-1.4
3.50	1.87	476	2.46	.740	"	-1.50	470	-1.2
4.38	2.09	527	2.72	.735	"	-0.75	525	-0.4

APPENDIX "D"

DATA

TABLE 20



WATER

FLOW COEFFICIENT DATA
OPENINGS 26-6

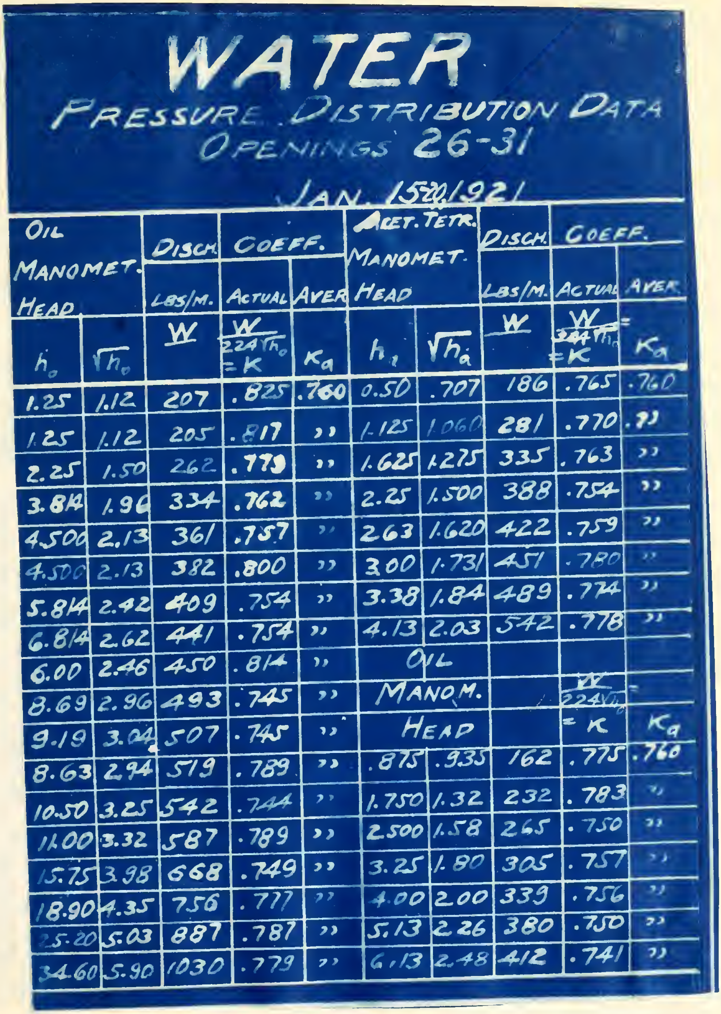
JAN 26, 1921

ACET. TETR. MANOM. HEAD		DISCH.		COEFFICIEN.		COEFF. % ERROR	CALCUL. DISCHAR.	FULL LOAD % ERROR
		LBS/M.	VEL.	ACTUAL	AVER.			
h_o	$\sqrt{h_o}$	W	$.00516W = V$	$\frac{W}{344\sqrt{h_a}} = K$	K_a	$100\frac{K-K_a}{K_a}$	$344K_a\sqrt{h_a} = W_1$	$100\frac{W_1-W}{500}$
.50	.707	171.25	.88	.705	.730	+3.50	178	+1.40
.813	.900	217	1.12	.704	"	+3.50	226	+1.60
1.814	1.347	334.3	1.73	.723	"	+1.00	338	+0.80
1.94	1.393	346	1.78	.723	"	+1.00	350	+0.80
2.31	1.517	371	1.91	.713	"	+2.00	381	+2.00
2.69	1.64	398	2.05	.707	"	+3.00	412	+2.80
2.81	1.68	418	2.15	.725	"	+0.50	421	+0.70
3.00	1.73	428	2.21	.722	"	+1.00	434	+1.00
3.20	1.79	442	2.28	.721	"	+1.00	449	+1.00
3.38	1.83	469	2.42	.734	"	-1.00	462	-1.00
3.44	1.85	468	2.41	.736	"	-1.00	465	-0.50
3.81	1.95	489	2.52	.730	"	0	490	0
4.13	2.03	514	2.65	.737	"	-1.00	510	-1.00
4.25	2.06	525	2.71	.743	"	-1.50	518	-1.50
4.63	2.15	542	2.80	.733	"	-0.50	540	-0.50
4.88	2.21	557	2.88	.735	"	-0.75	555	-0.75

APPENDIX "D"

DATA

TABLE 21



WATER

PRESSURE DISTRIBUTION DATA
OPENINGS 26-31

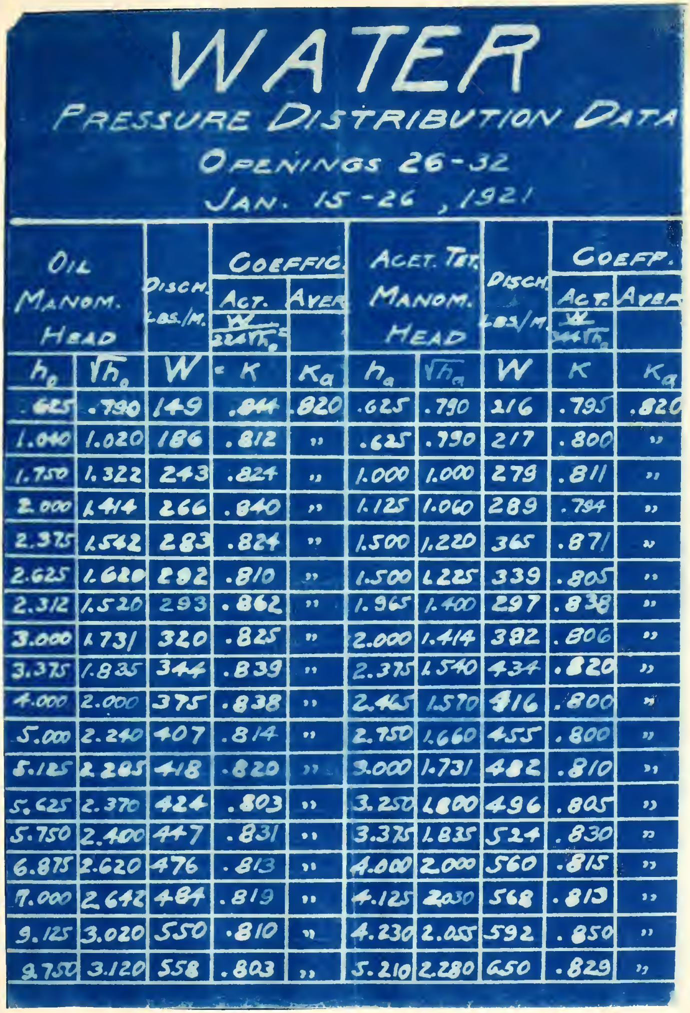
JAN. 15th 1921

OIL MANOMET. HEAD		DISCH.	COEFF.		ACET. TETR. MANOMET. HEAD		DISCH.	COEFF.	
		LBS/M.	ACTUAL	AVER.			LBS/M.	ACTUAL	AVER.
h_o	$\sqrt{h_o}$	W	$\frac{W}{224\sqrt{h_o}} = K$	K_a	h_a	$\sqrt{h_a}$	W	$\frac{W}{224\sqrt{h_a}} = K$	K_a
1.25	1.12	207	.825	.760	0.50	.707	186	.765	.760
1.25	1.12	205	.817	"	1.125	1.060	281	.770	"
2.25	1.50	262	.779	"	1.625	1.275	335	.763	"
3.814	1.96	334	.762	"	2.25	1.500	388	.754	"
4.500	2.13	361	.757	"	2.63	1.620	422	.759	"
4.500	2.13	382	.800	"	3.00	1.731	451	.780	"
5.814	2.42	409	.754	"	3.38	1.84	489	.774	"
6.814	2.62	441	.754	"	4.13	2.03	542	.778	"
6.00	2.46	450	.814	"	OIL				
8.69	2.96	493	.745	"	MANOM.			$\frac{W}{224\sqrt{h_o}}$ =	
9.19	3.04	507	.745	"	HEAD			= K	K_a
8.63	2.94	519	.789	"	.875	.935	162	.775	.760
10.50	3.25	542	.744	"	1.750	1.32	232	.783	"
11.00	3.32	587	.789	"	2.500	1.58	265	.750	"
15.75	3.98	668	.749	"	3.25	1.80	305	.757	"
18.90	4.35	756	.777	"	4.00	2.00	339	.756	"
25.20	5.03	887	.787	"	5.13	2.26	380	.750	"
34.60	5.90	1030	.779	"	6.13	2.48	412	.741	"

APPENDIX "D"

DATA

TABLE 22.



WATER

PRESSURE DISTRIBUTION DATA

OPENINGS 26-32

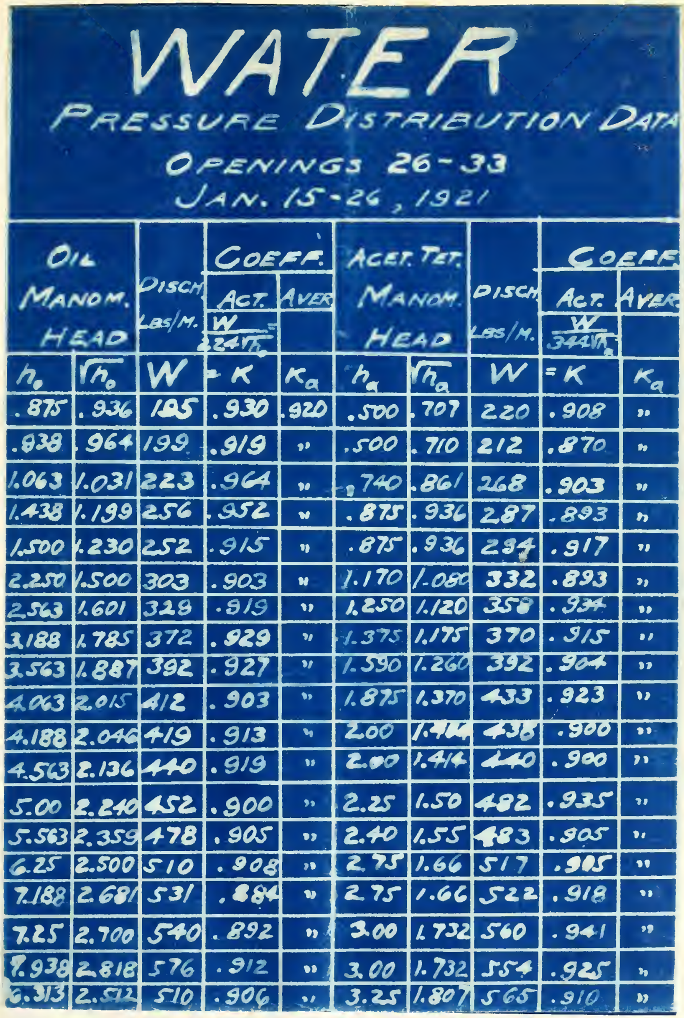
JAN. 15-26, 1921

OIL MANOM. HEAD		DISCH. LBS./M.	COEFFIC.		ACET. TET. MANOM. HEAD		DISCH. LBS./M.	COEFF.	
			ACT. $\frac{W}{324\sqrt{h_o}}$	AVER.				ACT. $\frac{W}{344\sqrt{h_a}}$	AVER.
h_o	$\sqrt{h_o}$	W	= K	K_a	h_a	$\sqrt{h_a}$	W	K	K_a
.625	.790	149	.844	.820	.625	.790	216	.795	.820
1.040	1.020	186	.812	"	.625	.790	217	.800	"
1.750	1.322	243	.824	"	1.000	1.000	279	.811	"
2.000	1.414	266	.840	"	1.125	1.060	289	.794	"
2.375	1.542	283	.824	"	1.500	1.220	365	.871	"
2.625	1.620	292	.810	"	1.500	1.225	339	.805	"
2.312	1.520	293	.862	"	1.965	1.400	297	.838	"
3.000	1.731	320	.825	"	2.000	1.414	382	.806	"
3.375	1.835	344	.839	"	2.375	1.540	434	.820	"
4.000	2.000	375	.838	"	2.465	1.570	416	.800	"
5.000	2.240	407	.814	"	2.750	1.660	455	.800	"
5.125	2.285	418	.820	"	3.000	1.731	482	.810	"
5.625	2.370	424	.803	"	3.250	1.800	496	.805	"
5.750	2.400	447	.831	"	3.375	1.835	524	.830	"
6.875	2.620	476	.813	"	4.000	2.000	560	.815	"
7.000	2.642	484	.819	"	4.125	2.030	568	.813	"
9.125	3.020	550	.810	"	4.230	2.055	592	.850	"
9.750	3.120	558	.803	"	5.210	2.280	650	.829	"

APPENDIX "D"

D A T A

T A B L E 23.



WATER

PRESSURE DISTRIBUTION DATA

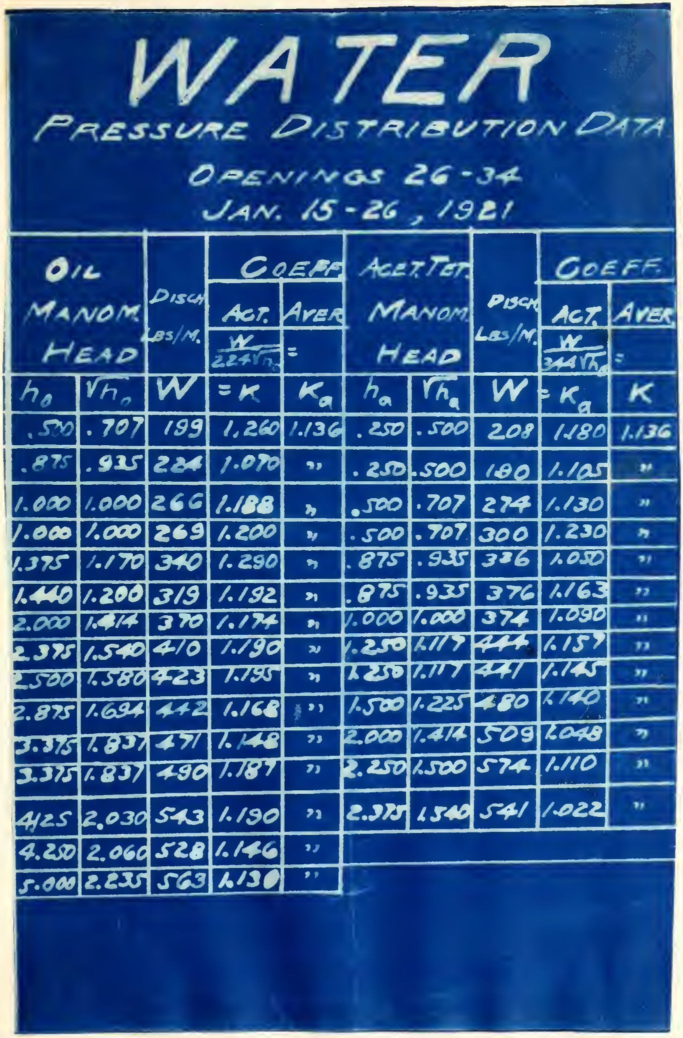
OPENINGS 26-33
JAN. 15-26, 1921

OIL MANOM. HEAD		DISCH LBS/M.	COEFF. ACT. $\frac{W}{224\sqrt{h_o}}$	COEFF. AVER.	ACET. TET. MANOM. HEAD		DISCH LBS/M.	COEFF. ACT. $\frac{W}{344\sqrt{h_a}}$	COEFF. AVER.
h_o	$\sqrt{h_o}$	W	= K	K_a	h_a	$\sqrt{h_a}$	W	= K	K_a
.875	.936	195	.930	.920	.500	.707	220	.908	"
.938	.964	199	.919	"	.500	.710	212	.870	"
1.063	1.031	223	.964	"	.740	.861	268	.903	"
1.438	1.199	256	.952	"	.875	.936	287	.893	"
1.500	1.230	252	.915	"	.875	.936	294	.917	"
2.250	1.500	303	.903	"	1.170	1.080	332	.893	"
2.563	1.601	328	.919	"	1.250	1.120	358	.934	"
3.188	1.785	372	.929	"	1.375	1.175	370	.915	"
3.563	1.887	392	.927	"	1.590	1.260	392	.904	"
4.063	2.015	412	.903	"	1.875	1.370	433	.923	"
4.188	2.046	419	.913	"	2.00	1.414	438	.900	"
4.563	2.136	440	.919	"	2.00	1.414	440	.900	"
5.00	2.240	452	.900	"	2.25	1.50	482	.935	"
5.563	2.359	478	.905	"	2.40	1.55	483	.905	"
6.25	2.500	510	.908	"	2.75	1.66	517	.905	"
7.188	2.681	531	.884	"	2.75	1.66	522	.918	"
7.25	2.700	540	.892	"	3.00	1.732	560	.941	"
7.938	2.818	576	.912	"	3.00	1.732	554	.925	"
6.313	2.512	510	.906	"	3.25	1.807	565	.910	"

APPENDIX "D"

DATA

TABLE 24.



WATER

PRESSURE DISTRIBUTION DATA

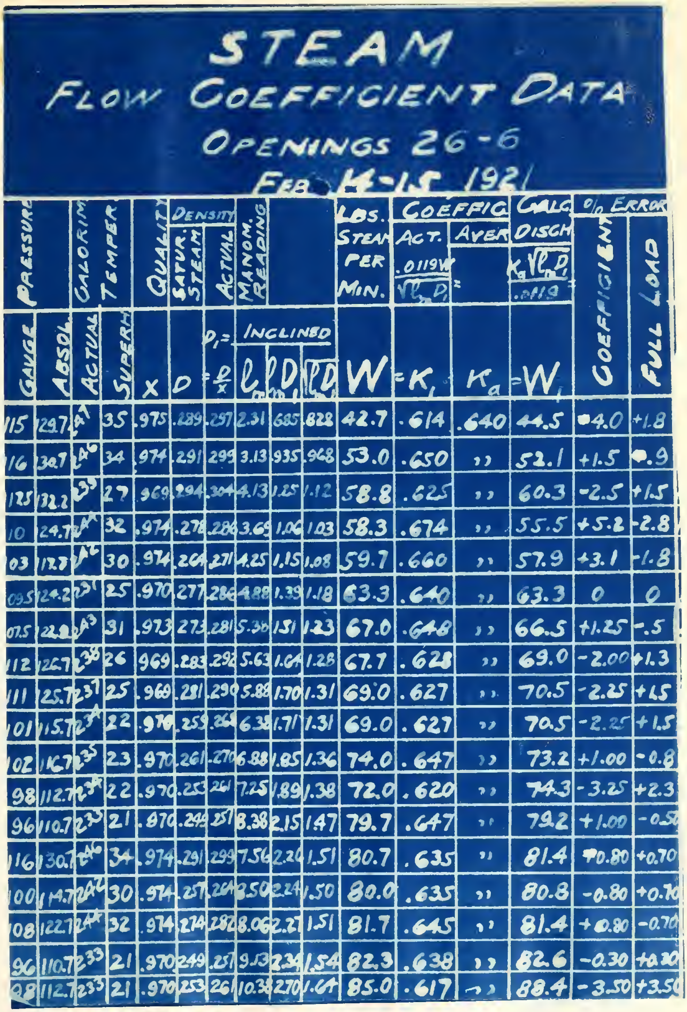
OPENINGS 26-34
JAN. 15-26, 1921

OIL MANOM. HEAD		DISCH LBS/M.	COEFF.		ACET.TET. MANOM. HEAD		DISCH LBS/M.	COEFF.	
			ACT.	AVER.				ACT.	AVER.
			$\frac{W}{224\sqrt{h_o}}$	=				$\frac{W}{344\sqrt{h_a}}$	=
h_o	$\sqrt{h_o}$	W	= K	K_a	h_a	$\sqrt{h_a}$	W	= K_a	K
.500	.707	199	1.260	1.136	.250	.500	208	1.180	1.136
.875	.935	224	1.070	"	.250	.500	180	1.105	"
1.000	1.000	266	1.188	"	.500	.707	274	1.130	"
1.000	1.000	269	1.200	"	.500	.707	300	1.230	"
1.375	1.170	340	1.290	"	.875	.935	336	1.050	"
1.440	1.200	319	1.192	"	.875	.935	376	1.163	"
2.000	1.414	370	1.174	"	1.000	1.000	374	1.090	"
2.375	1.540	410	1.190	"	1.250	1.117	444	1.157	"
2.500	1.580	423	1.195	"	1.250	1.117	441	1.145	"
2.875	1.694	442	1.168	"	1.500	1.225	480	1.140	"
3.375	1.837	471	1.148	"	2.000	1.414	509	1.048	"
3.375	1.837	490	1.187	"	2.250	1.500	574	1.110	"
4.125	2.030	543	1.190	"	2.375	1.540	541	1.022	"
4.250	2.060	528	1.146	"					
5.000	2.235	563	1.130	"					

APPENDIX "D"

DATA

TABLE 55



STEAM

FLOW COEFFICIENT DATA

OPENINGS 26-6

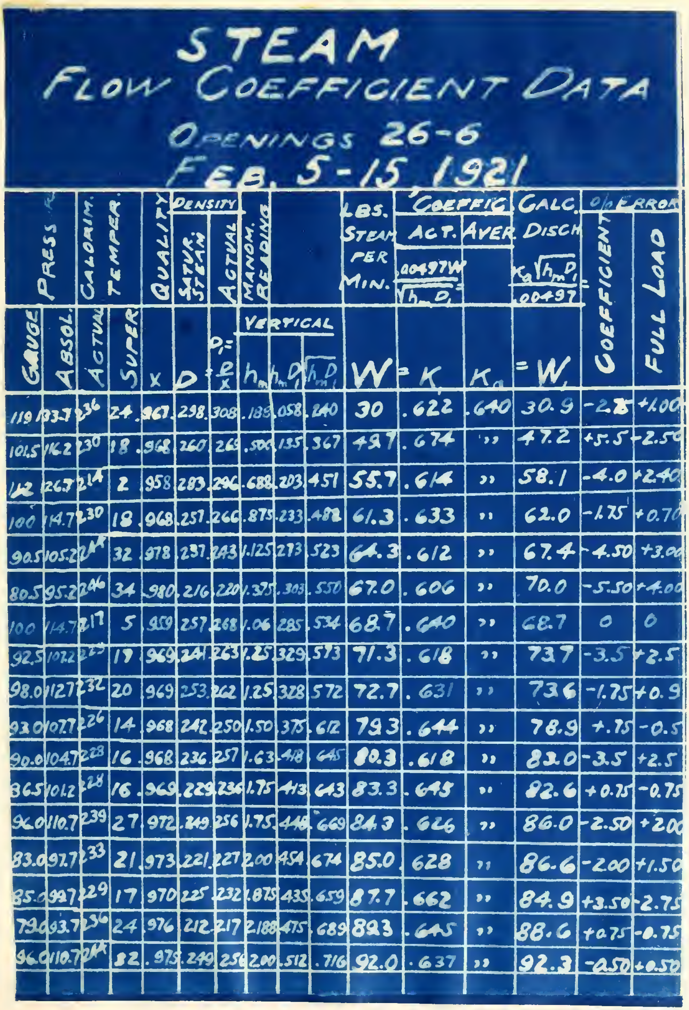
FEB 14-15 1921

PRESSURE GAUGE	ABSOL.	CALORIM ACTUAL	TEMPER SUPERH	QUALITY x	DENSITY SATUR. STEAM D	DENSITY ACTUAL $D_1 = \frac{D}{x}$	MANOM. READING INCLINED l_m	$l_m D_1$	$\sqrt{l_m D_1}$	LBS. STEAM PER MIN. W	COEFFIC ACT. $\frac{.0119 W}{\sqrt{l_m D_1}} = K_1$	COEFFIC AVER K_a	CALC DISCH $\frac{K_a \sqrt{l_m D_1}}{.0119} = W_1$	% ERROR COEFFICIENT	% ERROR FULL LOAD
115	129.7	247	35	.975	.289	.297	2.31	685	.828	42.7	.614	.640	44.5	-4.0	+1.8
16	130.7	246	34	.974	.291	.299	3.13	935	.968	53.0	.650	"	52.1	+1.5	-.9
125	132.2	239	27	.969	.294	.304	4.13	1.25	1.12	58.8	.625	"	60.3	-2.5	+1.5
10	24.7	244	32	.974	.278	.286	3.69	1.06	1.03	58.3	.674	"	55.5	+5.2	-2.8
03	112.7	242	30	.974	.264	.271	4.25	1.15	1.08	59.7	.660	"	57.9	+3.1	-1.8
09.5	124.2	231	25	.970	.277	.286	4.88	1.39	1.18	63.3	.640	"	63.3	0	0
07.5	122.2	243	31	.973	.273	.281	5.36	1.51	1.23	67.0	.648	"	66.5	+1.25	-.5
112	126.7	238	26	.969	.283	.292	5.63	1.64	1.28	67.7	.628	"	69.0	-2.00	+1.3
111	125.7	237	25	.969	.281	.290	5.88	1.70	1.31	69.0	.627	"	70.5	-2.25	+1.5
101	115.7	234	22	.970	.259	.268	6.38	1.71	1.31	69.0	.627	"	70.5	-2.25	+1.5
102	116.7	235	23	.970	.261	.270	6.88	1.85	1.36	74.0	.647	"	73.2	+1.00	-0.8
98	112.7	234	22	.970	.253	.261	7.25	1.89	1.38	72.0	.620	"	74.3	-3.25	+2.3
96	110.7	233	21	.970	.249	.251	8.38	2.15	1.47	79.7	.647	"	79.2	+1.00	-0.50
116	130.7	246	34	.974	.291	.299	7.56	2.20	1.51	80.7	.635	"	81.4	-0.80	+0.70
100	114.7	242	30	.974	.257	.264	8.50	2.24	1.50	80.0	.635	"	80.8	-0.80	+0.70
108	122.7	244	32	.974	.274	.282	8.06	2.27	1.51	81.7	.645	"	81.4	+0.80	-0.70
96	110.7	233	21	.970	.249	.257	9.13	2.34	1.54	82.3	.638	"	82.6	-0.30	+0.30
98	112.7	235	21	.970	.253	.261	10.38	2.70	1.64	85.0	.617	"	88.4	-3.50	+3.50

APPENDIX "D"

DATA

TABLE 26



STEAM
FLOW COEFFICIENT DATA
OPENINGS 26-6
FEB. 5-15, 1921

Press. Gauge	Press. Absol.	Calorim. Temper. Actual	Calorim. Temper. Super	Quality x	Density Satur. Steam D	Density Actual $D_1 = \frac{D}{x}$	Manom. Reading Vertical h_m	$h_m D_1$	$\sqrt{h_m D_1}$	Lbs. Steam per Min. W	Coeffic. Act. $\frac{.00497W}{\sqrt{h_m D_1}} = K_1$	Coeffic. Aver. K_a	Calc. Disch. $\frac{K_a\sqrt{h_m D_1}}{.00497} = W_1$	% Error Coefficient	% Error Full Load
119	133.7	236	24	.967	.298	.308	.185	.058	.240	30	.622	.640	30.9	-2.5	+1.00
101.5	116.2	230	18	.968	.260	.269	.506	.135	.367	49.7	.674	”	47.2	+5.5	-2.50
112	126.7	214	2	.958	.283	.296	.688	.203	.451	55.7	.614	”	58.1	-4.0	+2.40
100	114.7	230	18	.968	.257	.266	.875	.233	.482	61.3	.633	”	62.0	-1.75	+0.70
90.5	105.2	244	32	.978	.237	.243	1.125	.273	.523	64.3	.612	”	67.4	-4.50	+3.00
80.5	95.2	246	34	.980	.216	.220	1.375	.303	.550	67.0	.606	”	70.0	-5.50	+4.00
100	114.7	217	5	.959	.257	.268	1.06	.285	.534	68.7	.640	”	68.7	0	0
92.5	107.2	225	17	.969	.241	.263	1.25	.329	.573	71.3	.618	”	73.7	-3.5	+2.5
98.0	112.7	232	20	.969	.253	.262	1.25	.328	.572	72.7	.631	”	73.6	-1.75	+0.9
93.0	107.7	226	14	.968	.242	.250	1.50	.375	.612	79.3	.644	”	78.9	+.75	-0.5
90.0	104.7	228	16	.968	.236	.257	1.63	.418	.645	80.3	.618	”	83.0	-3.5	+2.5
86.5	101.2	228	16	.969	.229	.236	1.75	.413	.643	83.3	.645	”	82.6	+0.75	-0.75
96.0	110.7	239	27	.972	.249	.256	1.75	.448	.669	84.3	.626	”	86.0	-2.50	+2.00
83.0	97.7	233	21	.973	.221	.227	2.00	.454	.674	85.0	.628	”	86.6	-2.00	+1.50
85.0	99.7	229	17	.970	.225	.232	1.875	.435	.659	87.7	.662	”	84.9	+3.50	-2.75
79.0	93.7	236	24	.976	.212	.217	2.188	.475	.689	89.3	.645	”	88.6	+0.75	-0.75
96.0	110.7	244	32	.975	.249	.256	2.00	.512	.716	92.0	.637	”	92.3	-0.50	+0.50

APPENDIX "D"

DATA

TABLE 27.

STEAM

FLOW COEFFICIENT DATA

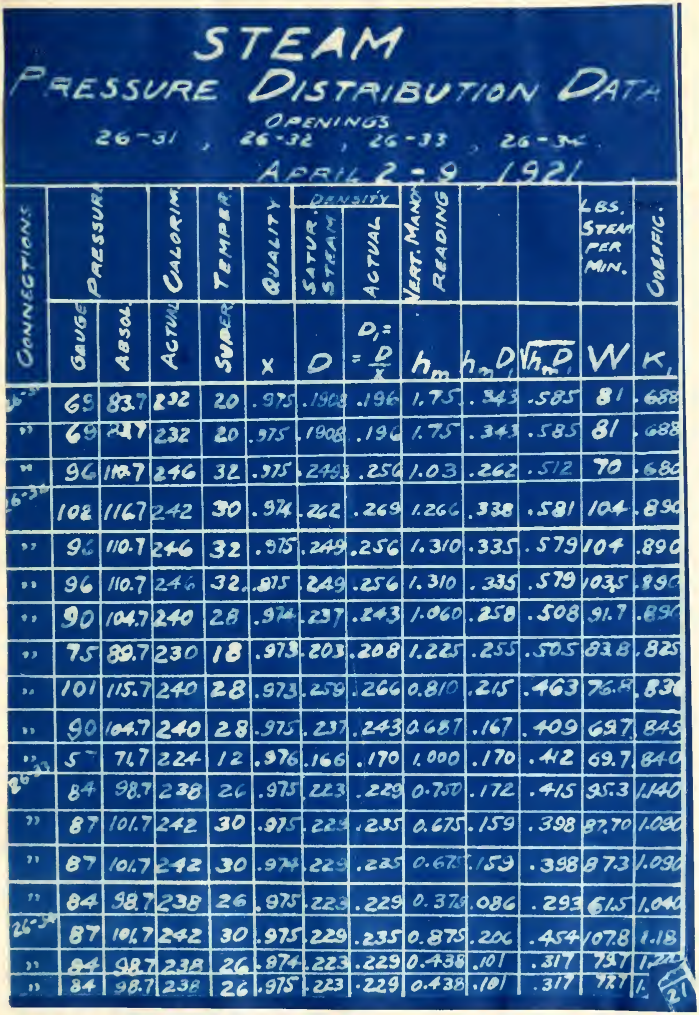
OPENINGS 26-6
FEB. 5-15 1921

PRESS. GAUGE	PRESS. ABSOL	CALOR. TEMPER. ACTUAL	CALOR. TEMPER. SUPER	QUALITY x	DENSITY SATUR. STEAM D	DENSITY ACTUAL $D_1 = \frac{D}{x}$	MANOM. READING h_m	$h_m D_1$	$\sqrt{h_m D_1}$	LBS. STEAM PER MIN. W	COEFF. ACT. $\frac{.0119W}{\sqrt{h_m D_1}} = K_1$	COEFF. AVER. K_a	CALC. DISCH. $\frac{K_a\sqrt{h_m D_1}}{.0119} = W_1$	% ERROR COEFFICIENT	% ERROR FULL LOAD
100	114.7	241	29	.973	.257	.264	9.88	2.61	1.62	86.7	.637	.640	87.2	-0.50	+0.50
96	110.7	233	21	.969	.249	.257	11.62	3.00	1.73	92.0	.634	"	93.4	-1.50	+1.40
104	118.7	242	30	.973	.266	.273	10.75	2.94	1.72	92.3	.639	"	92.6	-.15	+.15
102	116.7	242	30	.972	.261	.269	11.88	3.2	1.79	94.7	.629	"	96.5	-1.7	+1.7
92	106.7	232	20	.969	.240	.248	13.31	3.3	1.82	96.0	.628	"	98.0	-2.0	+2.0
101	115.7	242	30	.972	.259	.266	13.25	3.54	1.88	98.7	.625	"	101.1	-2.3	+2.3
ACET. TETRACHL.				x	D	D_1	h_a	$h_a D_1$	$\sqrt{h_a D_1}$	W	$\frac{.0229W}{\sqrt{h_a D_1}}$	K_a	$\frac{K_a\sqrt{h_a D_1}}{.0229}$		
113	127.7	269	57	.986	.285	.289	2.63	.760	.873	24.0	.630	.640	24.4	-1.5	+0.8
104	118.7	265	53	.986	.268	.268	3.94	1.06	1.03	28.4	.630	"	28.8	-1.5	+0.8
115	129.7	274	62	.986	.289	.294	3.94	1.16	1.08	29.7	.631	"	30.2	-1.75	+1.0
106	120.7	250	38	.976	.269	.276	6.06	1.67	1.30	34.5	.610	"	36.2	-4.75	+3.4
106.5	120.5	273	61	.990	.240	.242	6.31	1.53	1.24	35.5	.655	"	34.7	+2.25	-1.6
110.0	124.7	264	52	.985	.247	.251	8.19	2.05	1.43	39.6	.636	"	40.0	-0.80	+0.80
100.0	114.7	247	35	.976	.226	.242	9.75	2.35	1.54	43.0	.640	"	43.0	0	0
112	126.7	255	43	.978	.283	.289	8.31	2.40	1.55	41.7	.616	"	43.4	-3.75	+3.75
109	123.7	244	32	.974	.276	.284	9.25	2.62	1.62	43.7	.618	"	45.3	-3.50	+3.50
103	117.7	235	23	.969	.264	.272	9.88	2.69	1.64	45.0	.628	"	45.8	-1.6	+1.6
98	112.7	230	18	.968	.253	.262	12.0	3.14	1.77	47.7	.617	"	49.5	-3.5	+3.5

APPENDIX "D"

D A T A

T A B L E 28.



STEAM
PRESSURE DISTRIBUTION DATA
OPENINGS
26-31, 26-32, 26-33, 26-34
APRIL 2-9 1921

CONNECTIONS	PRESSURE		CALORIM.	TEMPER.	QUALITY	DENSITY		VERT. MANOM. READING			LBS. STEAM PER MIN.	COEFFIC.
						SATUR. STEAM	ACTUAL					
	GAUGE	ABSOL.	ACTUAL	SUPER.	x	D	$D_1 = \frac{D}{x}$	h_m	$h_m D_1$	$\sqrt{h_m D_1}$	W	K_1
26-3	69	83.7	232	20	.975	.1908	.196	1.75	.343	.585	81	.688
"	69	[illegible]	232	20	.975	.1908	.196	1.75	.343	.585	81	.688
"	96	110.7	246	32	.975	.2493	.256	1.03	.262	.512	70	.680
26-3	102	116.7	242	30	.974	.262	.269	1.266	.338	.581	104	.890
"	96	110.7	246	32	.975	.249	.256	1.310	.335	.579	104	.890
"	96	110.7	246	32	.975	.249	.256	1.310	.335	.579	103.5	.890
"	90	104.7	240	28	.974	.237	.243	1.060	.258	.508	91.7	.89
"	75	89.7	230	18	.973	.203	.208	1.225	.255	.505	83.8	.825
"	101	115.7	240	28	.973	.259	.266	0.810	.215	.463	76.8	.83
"	90	104.7	240	28	.975	.237	.243	0.687	.167	.409	69.7	.845
"	57	71.7	224	12	.976	.166	.170	1.000	.170	.412	69.7	.840
26-33	84	98.7	238	26	.975	.223	.229	0.750	.172	.415	95.3	1.140
"	87	101.7	242	30	.975	.229	.235	0.675	.159	.398	87.70	1.090
"	87	101.7	242	30	.974	.229	.235	0.675	.159	.398	87.3	1.090
"	84	98.7	238	26	.975	.223	.229	0.378	.086	.293	61.5	1.040
26-3	87	101.7	242	30	.975	.229	.235	0.875	.206	.454	107.8	1.18
"	84	98.7	238	26	.974	.223	.229	0.438	.101	.317	75.7	1.2
"	84	98.7	238	26	.975	.223	.229	0.438	.101	.317	72.7	1.

21

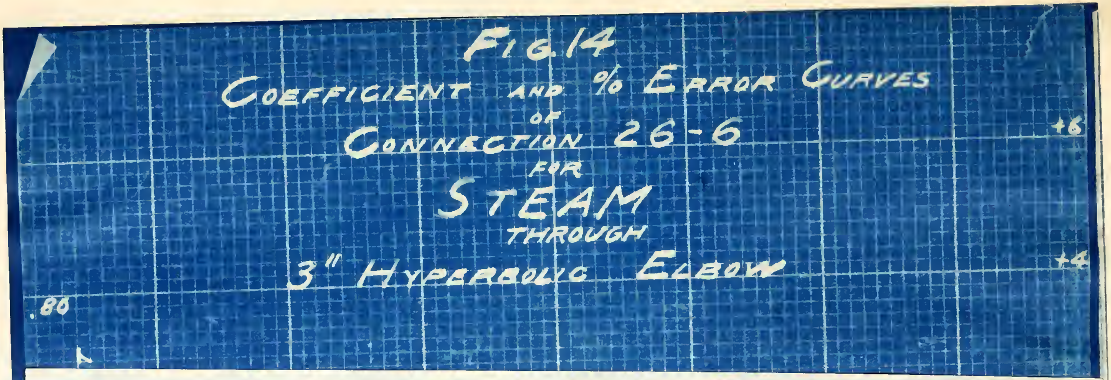


FIG. 14
COEFFICIENT AND % ERROR CURVES
OF
CONNECTION 26-6
FOR
STEAM
THROUGH
3" HYPERBOLIC ELBOW

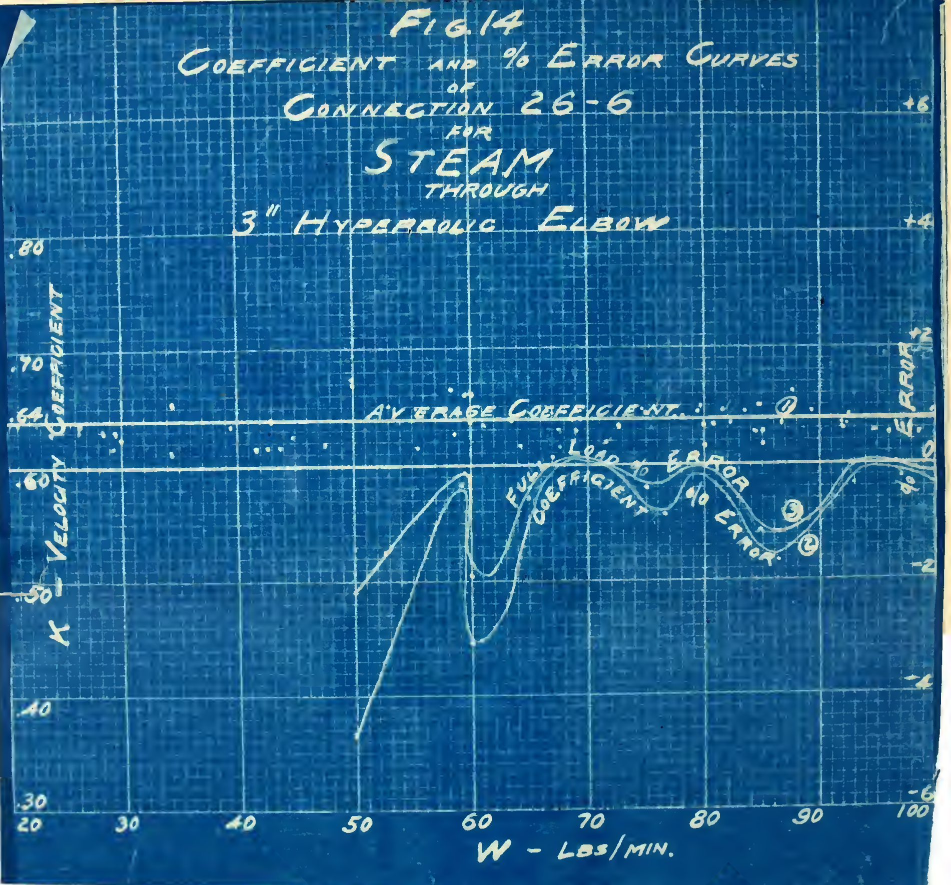
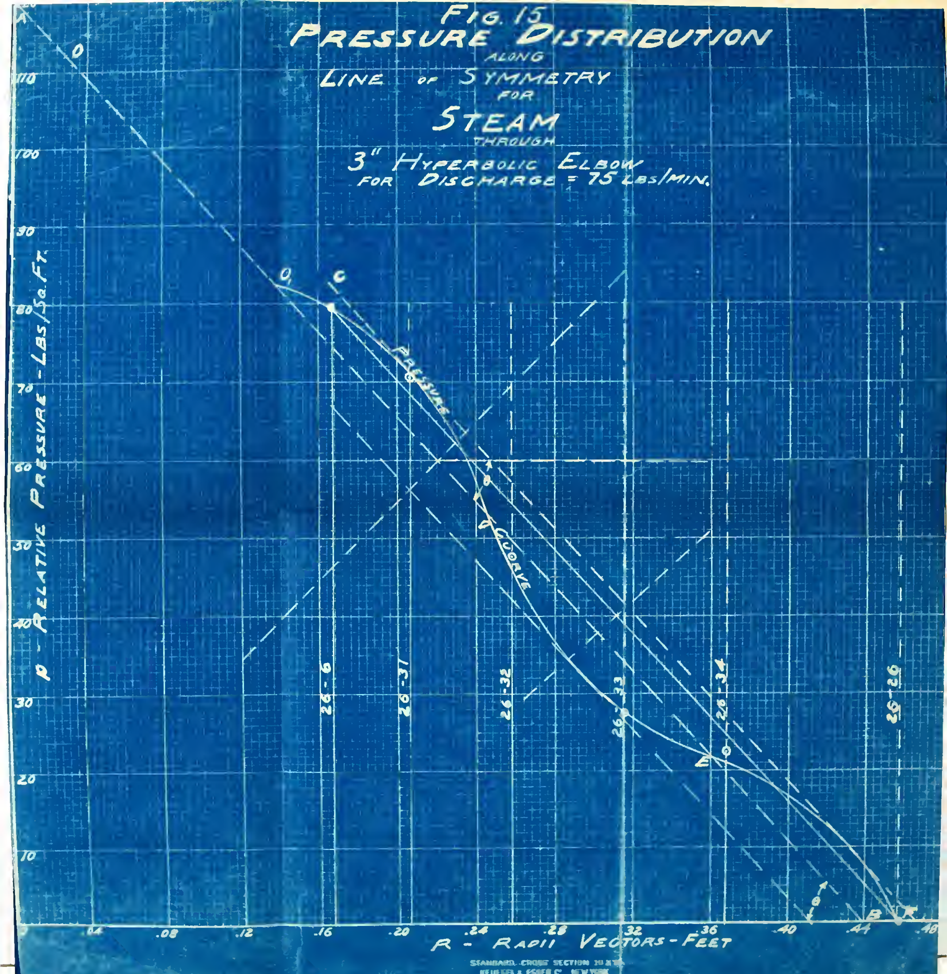


FIG. 14
COEFFICIENT AND % ERROR CURVES
OF
CONNECTION 26-6
FOR
STEAM
THROUGH
3" HYPERBOLIC ELBOW

FIG. 15
PRESSURE DISTRIBUTION
ALONG
LINE OF SYMMETRY
FOR
STEAM
THROUGH
3" HYPERBOLIC ELBOW
FOR DISCHARGE = 75 LBS/MIN.



STANDARD CROSS SECTION 20 X 20
KEUFFEL & ESSER Cº NEW YORK

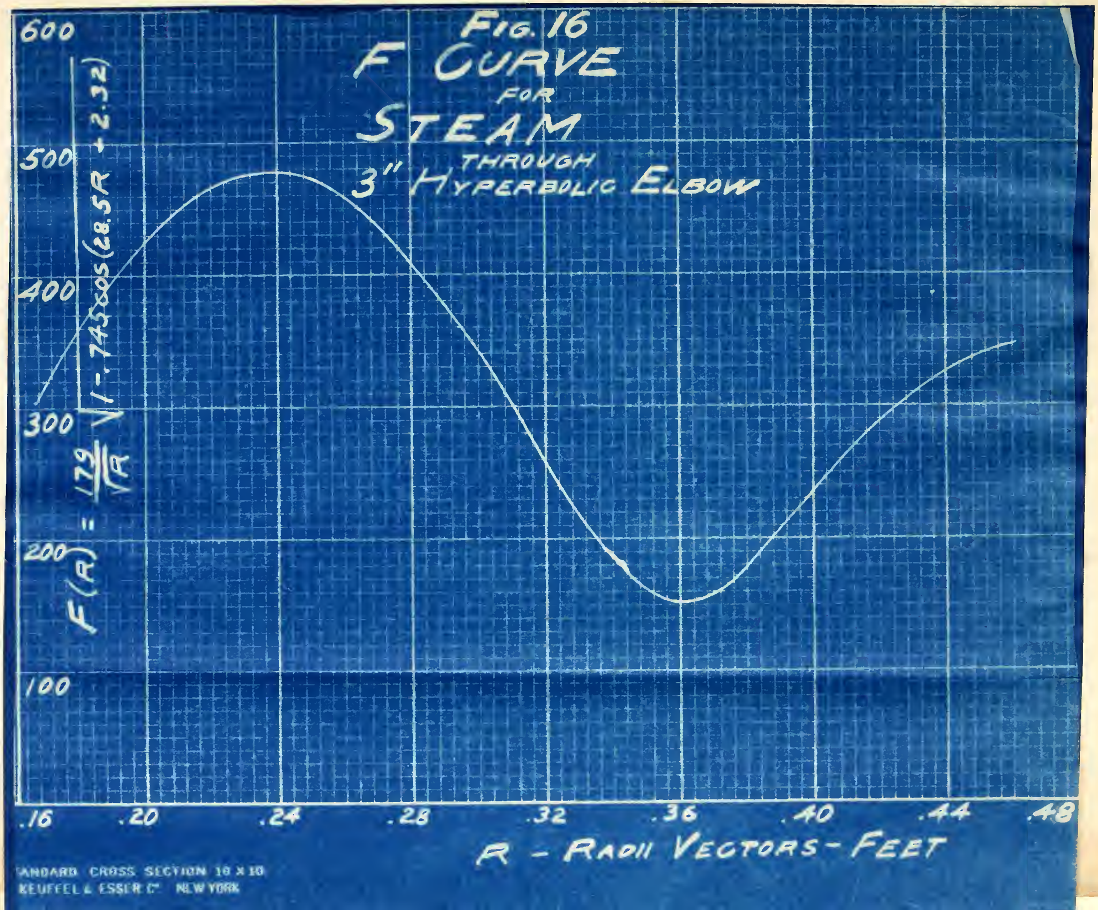


FIG. 16
F CURVE
FOR
STEAM
THROUGH
3" HYPERBOLIC ELBOW

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4. CONCLUSION 70 - 86

5. APPENDIX "A"
Elementary Theory of Plane
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DESIGN AND TEST OF AN HYPERBOLIC ELBOW FLOW METER

INTRODUCTION

1. It has been generally conceded during the last decade that flow in pipes is of a very complicated nature. Experiments made by Prof. Reynolds in England in the early nineties of the last century have shown the complexity of flow in even straight smooth glass tubes of sizes above capillary. He has demonstrated this fact by introducing coloring matter through a trumpet inside a glass tube where water was flowing. The particles of coloring matter first flowed in straight lines for short distances, but soon changed into curved paths the motion finally becoming turbulent, and all the water becoming colored. Photographs taken throughout these experiments illustrate nicely the change from stream line to turbulent flow, when the velocity of the water or size of the tube or both exceeded a certain definite limit. These limits were carefully noted by Prof. Reynolds and formulated. From an engineering point of view it is sufficient to remember that these limits are generally exceeded for all practical sizes of pipes of all ordinary velocities. When the comparative roughness of metal pipes and the complications of couplings, tees, ells, valves and other fittings are considered, the complexity of flow in pipes may be imagined.

2. We, therefore, conclude that for all

practical cases the motion in pipes is turbulent, the particles of fluid moving in all three dimensions. The nature of this motion is difficult of exact determination and varies continuously with change of loads, replacement of fittings, openings or closing of valves, etc. It is by visualizing these terse facts, that one appreciates the difficulty of measuring flow in pipes and places credence upon the often repeated statements that the so far prevailing methods of measuring flow by means of pitot tubes, orifices and venturi tubes are not as reliable as one may wish them to be.

3. If it were possible to arrange the measurement of flow so that the results would not be influenced by the nature of the piping, fittings, etc. of a given installation, we should have the first big step in the right direction. Thus, let us suppose that we break up the complex three dimensional motion in the pipe and change it for a short distance into a definitely known plane stream line flow. Evidently, if such a predetermined motion could be successfully secured regardless of what the motion in the pipe originally was, and if not until then were the flow to be measured, we should naturally expect great improvement in the reliability of results.

4. To accomplish this desirable change, in the case of a horizontal length of pipe, for example,

the motion in three dimensions is first reduced to plane horizontal motion by eliminating the vertical velocity component, i.e. by breaking up the up and down motion. A flange in front of an elbow in the pipe line is removed and replaced by a special flange into which are inserted a number of plane parallel "grids" or "fins" made of 1/32" monel metal plates with sharpened edges facing the flow. These "grids" are from 5"to 8" long and are placed in horizontal positions for the case considered. Any fluid particles striking the sharp edges of and the grids proper, at an angle, have their vertical motion broken up and are forced to move between the horizontal grids, although their plane motion may be very irregular.

5. Next, the plane motion is changed from irregular to stream line flow by introducing strong centrifugal forces in the form of an hyperbolic elbow, a tendency being created for the fluid particles to follow a definite stream line path.

6. Thus, instead of dealing with an unknown motion, we simplify the problem of flow measurement by reducing the motion to a definite and known condition, which can be easily analyzed, and whose existence can be checked experimentally. Even if stream

line motion were only *approached,* it will be conceded that this method is an appreciable step into the right direction as against taking the motion at random and assuming that the difference of pressure produced by the *indefinite* motion bears a *definite* relation to the average *velocity* in the pipe.

7. This thesis covers the theoretical and experimental investigation of the properties of an hyperbolic elbow when used for flow measurements. As will appear from the results of the tests on water and steam, the straightening grids and the hyperbolic elbow *actually* produce a *plane stream line* flow through the elbow. This is evidenced by the existence of a *definite* pressure distribution along certain lines of the elbow in exact agreement with a theoretically deduced law. If the motion were turbulent, the pressure distribution would change for each test and would follow no definite law of *any* kind, not saying anything about its agreement with a *theoretically deduced* law. The results of the tests show that the flow coefficient, K,of the relation,

$$v = K\sqrt{2g\,H}$$

bears a definite *predeterminable* relation to the dimensions of the elbow and to the nature of the fluid. The thesis, in addition, includes a brief

description of an electrical recording system used in conjunction with the hyperbolic elbow for flow measurement.

PHYSICAL ANALYSIS OF PLANE STREAM LINE FLOW

8. When the complex motion of a fluid in a pipe is reduced to plane stream line flow, it becomes possible to make a rational analysis of the flow problem without resorting to an empirical rules, coefficients, etc.

9. The practical engineer always demands a physical conception of any problem, while the mathematical physicist prefers to attack the problem from analytical point of view. There are certain cases, however, where the two methods of attack used jointly simplify the problem considerably. The undeveloped subject of hydraulics, or as the physicist calls it, hydrodynamics, would gain much in progress, if the engineer and mathematical physicist joined hands in attacking this difficult problem. For this reason, the writer has taken the liberty of giving in the appendices an analytical solution of the problem here considered, retaining its physical explanation in the paper proper.

10. The problem is:

Given a hyperbolic elbow of known di-

mensions with a known fluid flowing through the elbow in plane hyperbolic stream lines. Find the variation of pressure on the outside and inside curves A_1AA_2 and B_1BB_2 of the elbow. (See Fig. 1). Find the variation of velocity from point to point along the plane of symmetry ABCD, i.e., the plane cutting the elbow into two symmetrical halves. Finally, if the difference of pressure between any two points on the outside and the inside curves are known, what is the relation of the known pressure difference to the *average velocity* of the fluid passing through the elbow.

11. The solution of this problem is based upon two well known principles, viz.,

(1) The law of conservation of energy.

(2) The law of centrifugal force.

Thus, when a mass of fluid, weighing $\underline{W}$ lbs. is moving with a velocity of $\underline{v}$ ft./sec., it has

$$\frac{W}{2g} v^2 \text{ ft.-lbs.}$$

of kinetic energy. If this mass happens to occupy one cubic foot, $\underline{W}$ is the density of the fluid, and we may write:

1 cu. ft. contains $\frac{W}{2g} v^2$ ft.-lbs. of energy.

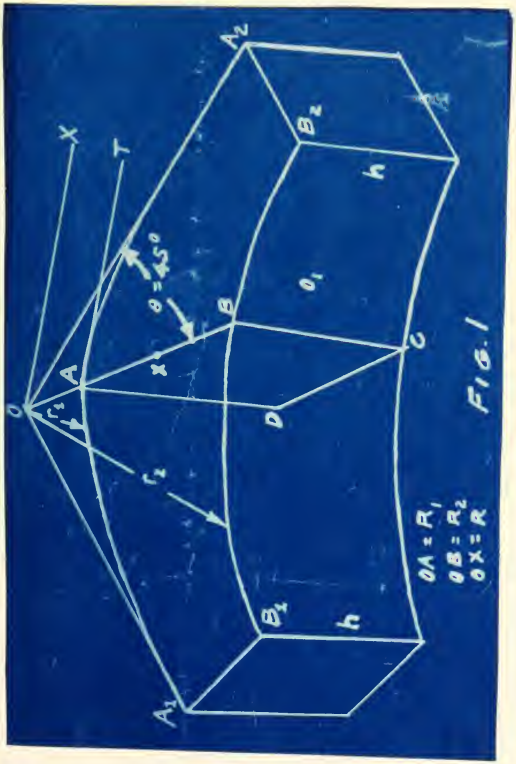


FIG. 1

13. Similarly, if this same cubic foot is under a pressure of P lbs./sq.ft., it has what may be designated as pressure potential energy, i.e., energy of the same nature as that possessed by a compressed spring. The energy for V cubic feet amounts to PV ft. -lbs., and for one cubic foot, it is evidently equal to P ft.-lbs. We may, therefore, write;

1 cu.ft. contains P ft.-lbs. of energy.

14. The total energy, E, possessed by this cubic foot of fluid is, therefore,

$$E = \frac{Wv^2}{2g} + P \quad (1)$$

The fluid may also have a variable potential energy of position if it moves between points at different levels. But if the motion is horizontal, this particular kind of energy remains constant and need not, therefore, be considered.

15. Now, let us consider a particle of fluid moving in stream lines without losing any energy by friction, or rather without changing any of its mechanical energy into heat energy. Then, the law of conservation of energy states, that the total energy of each cubic foot of the fluid does not change, as the particle moves along the stream line, i.e.,

$$E = \frac{W v^2}{2 g} + P = \frac{W v^2}{2 g} + P_1 \qquad (2)$$

16. Again, when a particle of fluid moves with a velocity of v ft./sec. along a curved stream line at a point where the radius of curvature is r ft., a centrifugal force,

$$f = \frac{W v^2}{g r} \text{ lbs.} \qquad (3)$$

will act on each cubic foot of fluid tending to force it to maintain its original direction of motion. Also, an equal centripetal force will have to be applied to the particle to oppose the centrifugal force and thus to make it possible for the particle to make the proper change in direction and to follow the curved line stream line. This opposing centripetal force is produced by variation of pressure along the normal to the stream line curve, the final pressure distribution being affected by the curvature of the wall or other restriction creating the curved motion.

17. Imagine now two parallel straight stream lines very close to each other, and in which the pressures and velocities of neighboring points of the two curves are equal. Now, if, at some point, the stream lines are suddenly bent into some curved shape, then, as shown in appendix A, there will have to occur a change in pressure along the common normal, r, of the two curves such that

$$\frac{dp}{dr} = -\frac{D v^2}{g r} \qquad (4)$$

i.e., the pressure variation will have to exactly balance the centrifugal force (compare equations 3 and 4) and will have to be equal and opposite to that force.

18. But, when the pressures, after originally having been equal change, then, since according to the law of conservation of energy,

$$\frac{W v_1^2}{2 g} + P_1 = \frac{W v_2^2}{2 g} + P_2 \quad (\text{see eq. } 2)$$

it is evident that the velocities, v_1 and v_2 must change also. And, when the velocity $\underline{v}$ at any point changes the centrifugal force,

$$f = \frac{D v^2}{g r} \qquad (\text{See eq. } 3)$$

must also change. The change of centrifugal force in turn requires a change in pressure distribution (see eq. 4) and this in turn requires a change in velocity (see eq. 2), etc.,

19. The point which it is desired to bring out is that pressure, velocity and centrifugal force are interlinked and that certain conditions must be satisfied before stream line motion can be established and maintained in equilibrium. It is conceivable to choose such a path that it would be impossible to satisfy the relation between pressure, velocity

and centrifugal force, and, as a result, the particles would not move in stream lines but would have a swirling motion instead.

20. It can be shown analytically, (see Lamb's Hydrodynamics p. 69-70) that in order to maintain plane stream line frictionless flow, the curved path must fall within a certain class of curves, whose general equation in polar co-ordinated is of the form,

$$r^n \sin n\theta = \text{constant} \qquad (5)$$

21. For $n = 1$ and $n = 2$, this equation takes the forms,

$$r\sin\theta = \text{constant} \qquad (6)$$

$$r^2 \sin 2\theta = \text{constant} \qquad (7)$$

and in rectangular co-ordinates,

$$y = \text{constant} \qquad (7a)$$

$$r^2\sin^2\theta = 2r\sin\theta\, r\cos\theta, \text{ or}$$

$$xy = C_1 \qquad (7a)$$

22. The first curve is a straight line, while the second curve is a rectangular hyperbola. Evidently, a circle will not satisfy the conditions required by equation (5). That is the reason why a hyperbolic and not a circular elbow was chosen for flow measurement. Since the Sargent Steam Meter Co., the owners of the hyperbolic elbow rights, are also in control of Avid Levine's circular elbow (for description see Proc. A.S.M.E., April 1914), the choice

is caused for purely scientific and not commercial reasons.

23. Without going into the details found in the appendices, it may be remarked that if a particle of fluid moves tangent to any curved path, i.e., along this path, its velocity components along the x-axis and the y-axis must bear such relation to each other that the direction of the resultant velocity should be tangent to the curved stream line. Therefore, if we note the fact that the tangent of the angle made by the tangent to any point of a rectangular hyperbola,

$$xy = c_1$$

can be expressed by,

$$\tan \theta = \frac{dy}{dx} = -\frac{y}{x} \qquad (18)$$

it will appear somewhat clearer why the x and y velocity components can be expressed by,

$$v_x = - Bx \qquad (9)$$

$$v_y = By \qquad (10)$$

Thus, from Fig.2,

and from the last equarions,

$$\tan \theta = \frac{v_y}{v_x} = -\frac{y}{x} \qquad (11)$$

which agrees with eq.8, showing that the resultant velocity,

v, is tangent to the curve as it should. This resultant velocity may evidently be expressed by:

$$v = \sqrt{v_x^2 + v_y^2} =$$

$$= B\sqrt{x^2 + y^2} =$$

$$= Br. \qquad (12)$$

The formulae for the velocities are based upon the stream lines being composed of a family of rectangular hyperbolas of the general form

$$xy = C$$

the value of the constant C for each hyperbola depending upon its shortest radius vector, R. In fact,

$$C = -\frac{R^2}{2} \qquad (13)$$

for each hyperbola.

24. It is shown in the appendices that for frictionless flow, the difference of pressure between any two points whose radii vectors are r_1 and r_2 respectively, (see Fig.1), can be expressed by

$$P_1 - P_2 = \frac{DB^2}{2g} (r_2^2 - r_1^2) \qquad (14)$$

D = density of the fluid, lbs./cu.ft.

B = constant (same as in eq.9,10 & 12)

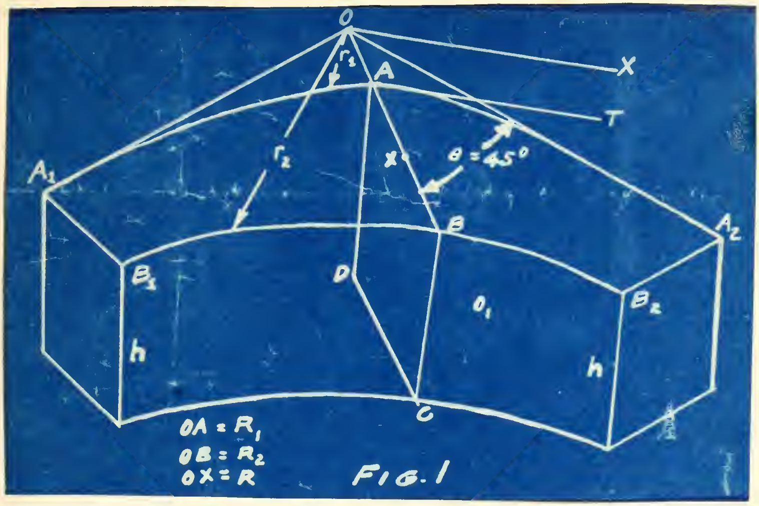

O
X
T
A
r_1
r_2
x
$\theta = 45^0$
A_1
A_2
B
B_1
B_2
D
O_1
h
h
C
$OA = R_1$
$OB = R_2$
$OX = R$

FIG. 1

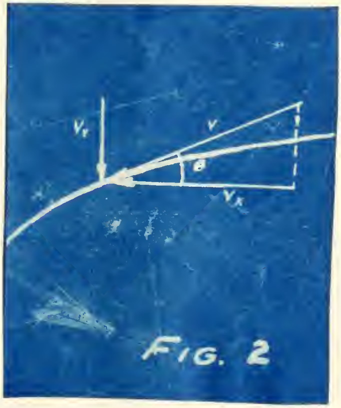

V_r
V
θ
V_x
x

FIG. 2

25. Furthermore, in any horizontal plane, say $A_1\ AA_2 - B_2\ BB_1$ (see Fig.1), all the fluid particles must cross the line of symmetry $OABO_1$. Also, the direction of flow of these particles when crossing the line $\underline{OABO_1}$ must be along the tangents to the family of hyperbolas at their points of intersection with the line $OABO_1$. Since at these points, $x = -y$, we have,

$$\tan \theta = \frac{-y}{x} = \frac{x}{x} = 1 \qquad \text{(see 8)}$$

or all these tangents and therefore, also the velocities are perpendicular to the line $OABO_1$. ($\theta = 45^o$; angle $XOA_2 = 45^o$; or angle $XOBB_1 = 90^o$). Thus in Fig. 1, the tangent $\underline{TA}$ to the outside hyperbola $A_1\ AA_2$ is parallel to the line OX, and therefore, is perpendicular to the line of symmetry $OABO_1$. The same is true for the rest of the hyperbolas.

26. Now the velocity at any point along the line of symmetry can be expressed by

$$v = BR \quad \text{(see eq.10)}$$

or is proportional to the shortest radius vector, $\underline{R}$, of any hyperbola considered. Therefore, it is evident that the average velocity is the arithmetical mean of the maximum and the minimum velocities at the internal and external hyperbolas respectively; i.e.,

$$V_a = B\frac{\left(R_1 + R_2\right)}{2} \qquad (15)$$

(See Fig. 1)

27. The vertical cross-section of the hyperbolic elbow is made rectangular instead of circular and of uniform depth $BC = h$ (See Fig 1.), the object being to avoid the complications of variable horizontal cross-sections and to aid the uniformity of flow in the different horizontal planes of the elbow. Therefore, for frictionless flow, the average velocity through the line of symmetry $OABO_1$ is the same as that through the plane of symmetry, ABCD, its value being given in eq.15.

28. Rewriting expressions for pressure difference and average velocity, viz.,

$$P_1 - P_2 = \frac{DB^2}{2g}\left(r_2^2 - r_1^2\right) \quad \text{(See eq.14)}$$

and

$$V_a = B\frac{\left(R_1 + R_2\right)}{2} \qquad \text{(See eq.15)}$$

we note that each of them involves the unknown constant B. From the last equation (15) it is evident that this constant is proportional to the average velocity and therefore, also to the rate of flow. From equation (14) it is evident that the difference of pressure is proportional to the square of the constant B and therefore, to the square of the rate of flow.

That is the reason why in the case of the hyperbolic elbow, as in the case of other flow matering devices, the average velocity or rate of flow is proportional to the square foot of the difference of pressure between shy two points on the external and internal curves of the elbow.

29. Mathematically, it is easily shown that,

$$v_a = \sqrt{\frac{(R_1+R_2)^2}{4(r_2^2-r_1^2)}}\sqrt{2g\frac{(P_1-P_2)}{D}}$$

$$= K\sqrt{2gH} \qquad (16)$$

(see Appendix A)

where,

$$K = \sqrt{\frac{(R_2+R_1)^2}{4(r_2^2-r_1^2)}} \qquad (17)$$

and depends for its value upon the dimensions R_1, R_2 of the elbow, and upon the points r_1, r_2 where the pressure connections are made. For the special case, when pressure connections are made at the center where,

$r_1 = R_1$ and $r_2 = R_2$,

the constant K becomes

$$K = \sqrt{\frac{(R_2+R_1)^2}{4(R_2^2-R_1^2)}} = \sqrt{\frac{R_2+R_1}{4(R_2-R_1)}} \qquad (18)$$

The simplicity and definiteness of the relation between average velocity and pressure difference, as shown in equation (16), make the rectangular hyperbolic elbow admirably suited for flow measurement.

30. In the case of flow with friction, it can be shown that instead of the velocity being directly proportional to the radius vector, as given by,

$$v = Br \text{ (eq.12)}$$

the velocity, for any given radius vector, r, varies as we pass on from one stream line to another, i. e., v is a function of the stream line as well as of the radius vector r. Thus, we may write

$$\begin{aligned} v &= r\,F_1(-xy) \\ &= r\,F(R) \qquad\qquad (19) \end{aligned}$$

where R is a characteristic of any hyperbola (see eq. 13), being its shortest radius vector.

31. The same unknown function, F (R), is involved in the equations for the pressure, **P**, at any point. Because of the difficulty, if not the impossibility of solving the differential equations of motion in the new form they take for flow with friction, (see their form in the Appendix B) the experimental engineer steps in where the mathematical physicist leaves off

and discovers the fact that the unknown function, F (R), happens to be involved in the expression for pressure distribution along the line of symmetry, $x = -y$. Thus, as shown in the appendix, the following equation holds for the line of symmetry,

$$\frac{dp}{dr} = -\frac{D}{g} RF^2 \qquad (20)$$

32. Therefore, by experimentally finding the pressures, at different points of the line of symmetry $x = -y$, the pressure distribution curve for this line may be plotted, and we then get the curve,

$$p = f(R) \qquad (21)$$

For frictionless flow, the pressure distribution curve is a parabola or a curve of the second degree, and its first derivative, $\frac{dp}{dr}$, is a straight line or proportional to the radius vector, R. Thus, as shown in the Appendix A,

$$p = p_0 - \frac{DB}{2\,g}\,{}^2R^2 \qquad (22)$$

and

$$\frac{dp}{dr} = -\frac{DB^2R}{g} =$$

$$= -\frac{DRB^2}{g} = -\frac{DRF^2}{g} \qquad (23)$$

(compare with eq 20) which shows that for frictionless flow,

$$F = B \qquad (24)$$

Therefore, if values of F were plotted against those of radii vectors, R, we should get for

frictionless flow a straight horizontal line. If the same were done from the actual experimental data as given by the pressure distribution curve, the nature of the F curve would indicate the extent of deviation of the actual flow from frictionless, flow. For example, from the results on water flowing through a 3" pipe, it was found that the actual F curve is a straight inclined line; i.e.,

$$F = B \quad (1 + A_1R) \qquad (25)$$

33. After the nature of the F curve is once found experimentally, the expressions for velocity and pressure can at once be written down. Thus, along the line of symmetry,

$$v = RF = RB \quad (1 + A_1B)$$

$$= B\ (R + A_1R^2) \qquad (26)$$

and from eq. (20),

$$P = p_o - \int \frac{D}{g} RF^2 dr =$$

$$= p_o - \frac{D}{g} \int RB^2 (1 + A_1R)^2 = \qquad (27)$$

(for exact form and details)
(see Appendix A)

Similarly, the average velocity is found to be,

$$v_a = \frac{\int_{R_1}^{R_2} v\,dR}{R_2 - R_1} = \frac{\int_{R_1}^{R_2} B(R + A_1R^2)dR}{R_2 - R_1}$$

$$= \quad B\theta \, (R_1, R_2) \qquad (28)$$

and the difference of pressure between the centers of the external and internal curves becomes,

$$p_1 - p_2 = \frac{B^2 D}{g} \left[f\,(R_2) - f\,(R_1) \right] \qquad (29)$$

34. Comparing the last expressions for average velocity and pressure difference with those for friction-less flow (see eq. 14 and 15) we note that, as before, B is proportional to the average velocity or rate of flow, while the pressure difference is proportional to the square of B, or to the square of the average velocity. The only difference is the nature of the functions of R involved in the expressions for flow with friction. Similarly, the relation between the average velocity and pressure difference for flow with friction takes the form,

$$v_a = \sqrt{\frac{\theta\,(R_2\,, R_1)^2}{\left[\frac{f(R_2)}{2} - \frac{f(R_1)}{2}\right]}}\,\sqrt{2g\,H} =$$

$$= K\sqrt{2gH} \qquad (30)$$

where, as before, K is a function of the dimensions R_1 and R_2 of the hyperbolic elbow. Only the function in this case is of higher order than the comparatively simpler one for flow without friction (compare with eq.16).

35. We summarize the physical analysis of plane stream line flow by stating that the application of the law of conservation of energy and of the law of centrifugal force to plane hyperbolic stream flow yields the following results:

For frictionless flow:

$$v_x = -Bx \qquad \text{(see eq.9)}$$

$$v_y = By \qquad \text{(" " 10)}$$

$$v = Br \qquad \text{(" " 12)}$$

$$\frac{dp}{dR} = -\frac{D}{g} RB^2$$

$$V_a = B \frac{(R_1 + R_2)}{2} \qquad \text{(see eq. 15)}$$

$$p = p_0 - \frac{DB^2}{2g} r^2 \qquad \text{(see eq. 23)}$$

$$p_1 - p_2 = \frac{DB^2}{2g} (r_2^2 - r_1^2) \qquad \text{(see eq.14)}$$

$$v_a = K \sqrt{2gH} =$$

$$= \sqrt{\frac{(R_2 + R_1)^2}{4(r_2^2 - r_1^2)}} \sqrt{2gH} \qquad \text{(see eq.16)}$$

$$K = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}} ; \qquad \text{(see eq. 18)}$$

for $r_1 = R_1$ and $r_2 = R_2$;

$$F = B$$

For flow with friction.

$$v_x = -xF$$

$$v_y = yF$$

$$v = rF \qquad \text{(see eq.19)}$$

$$\frac{dp}{dR} = -\frac{D}{g} RF^2 \qquad \text{(see eq. 20)}$$

$$v_a = B\theta \quad (R_1, R_2)$$

$$p = p_0 - \frac{B^2 D}{g} f(R) \quad \text{(see eq. 27)}$$

$$p_1 - p_2 = \frac{B^2 D}{g} \left[f(R_2) - f(R_1) \right] \quad \text{(see eq.29)}$$

$$v_a = \sqrt{2gH} =$$

$$= \sqrt{\frac{\theta\,(R_2, R_1)^2}{4\left[\frac{f(R_2)}{2} - \frac{f(R_1)}{2}\right]}} \sqrt{2gH} \quad \text{(see eq.30)}$$

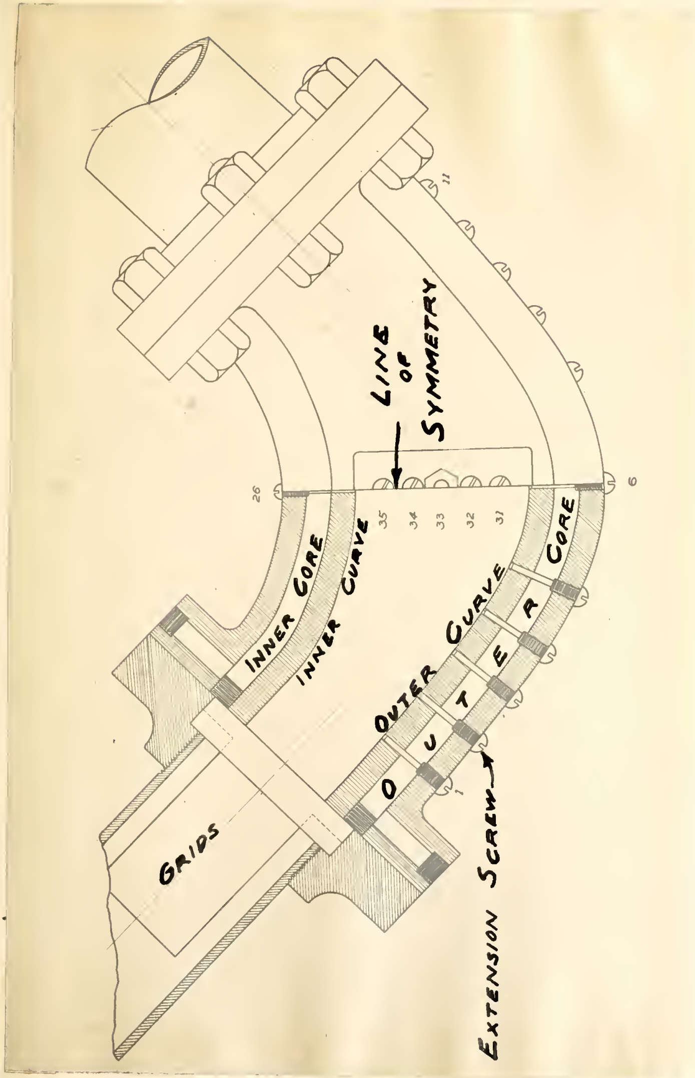
$$F(R) = B\,(1 + A_1 R + A_2 R^2 + \ldots\ldots\ldots\ldots)$$

EXPERIMENTAL DETERMINATION OF THE F.CURVE FROM THE PRESSURE DISTRIBUTION CURVE

APPARATUS AND ITS USE

36. A three inch long sweep elbow and the adjoining flanges were removed from a 3" water line at Armour Institute of Technology and replaced by the hyperbolic elbow with special pipe flanges shown in Figures 3 and 4.

Fig. 3



PLAN VIEW OF A 3" HYPERBOLIC ELBOW

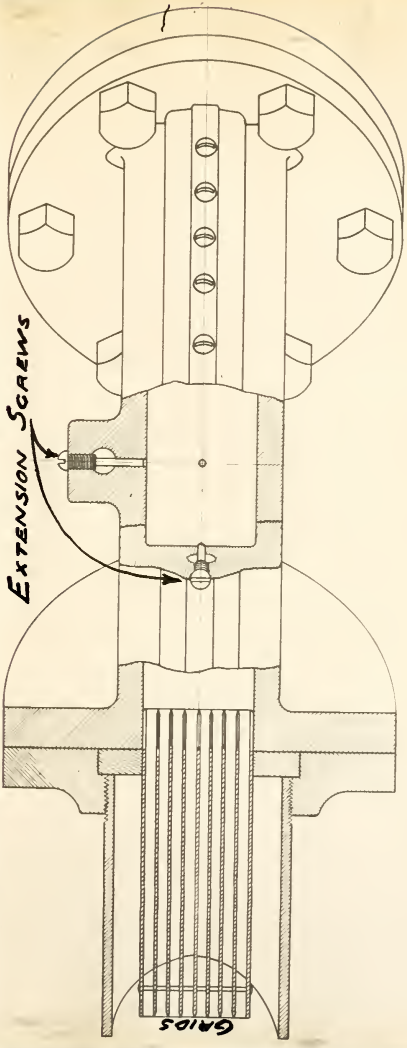


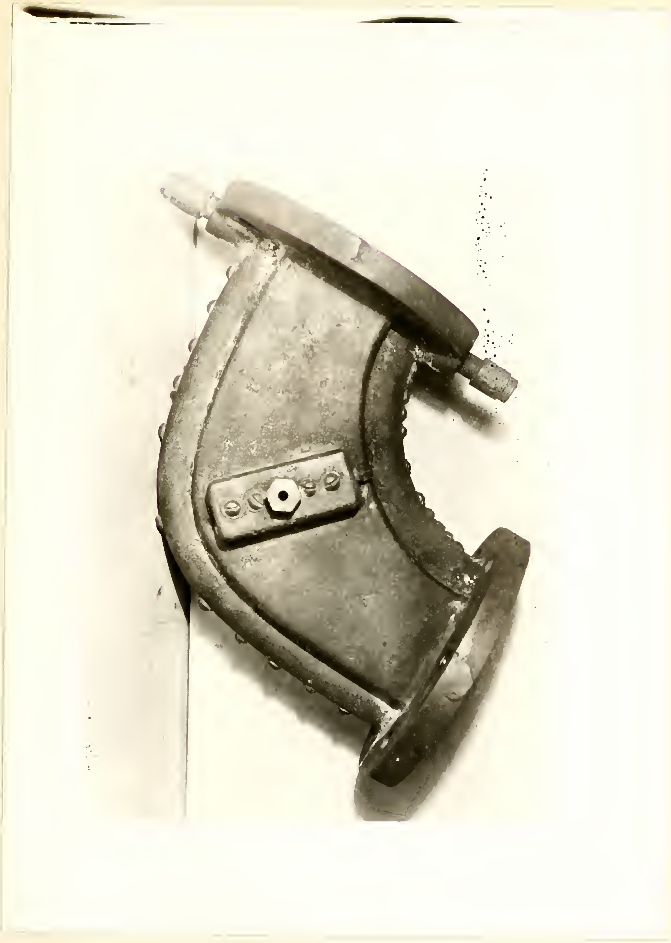
FIGURE 4.

REAR VIEW OF A 3" HYPERBOLIC ELBOW LOOKING FROM THE BACK

Into the flange, facing the flow, were inserted a number of sharp edged plane grids as shown in Fig. 4. All along the outer and inner curve a number of openings were made, each fitted with an extension screw. When the pressure at any point say 1, on the outer curve, was to be compared with that of any point, say 26, on the inner curve, the extension screws were taken out from openings 1 & 26 and replaced by ordinary screws without any extensions. This caused the pressures in the outside and inside cores to correspond with those at points 1 and 26, respectively, and the pressure difference between these two points was transmitted to and read off on a manometer.

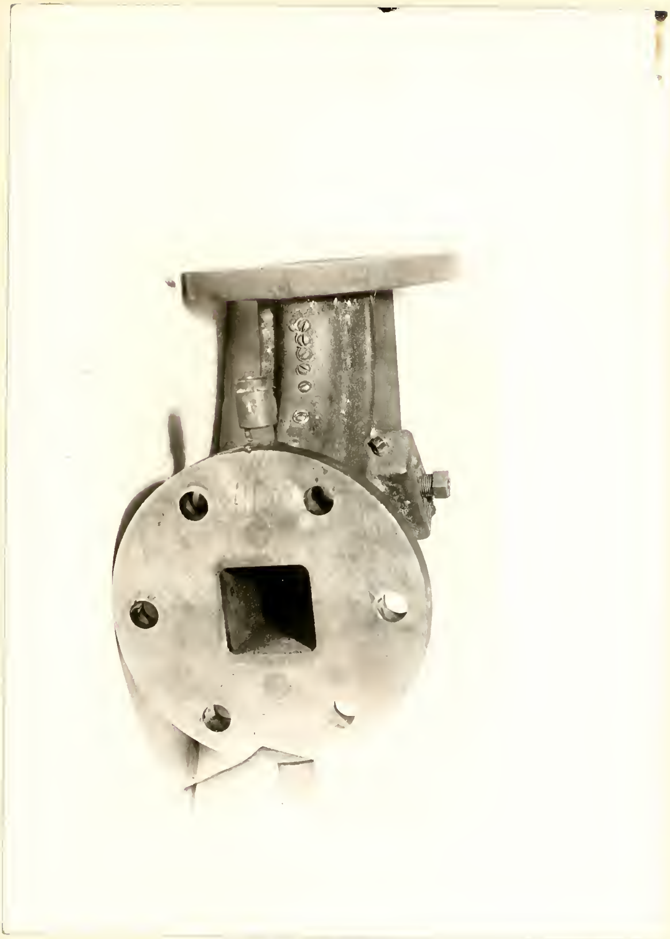
37. Similar openings were fitted with similar screws on the top of the elbow along the line of symmetry. The top core (see Fig 4) was made so that it could be connected to either the inner or outer core. The numbering of the openings is illustrated in Fig 3 and has been followed throughout the investigation. In this particular test, only the pressure distribution along the line of symmetry and the main coefficients were investigated for both water and steam, the analysis of pressure distribution along the external and internal curves having been left for a future date.

FIGURE 5



PHOTOGRAPH OF A 3" HYPERBOLIC ELBOW LOOKING AT THE TOP.

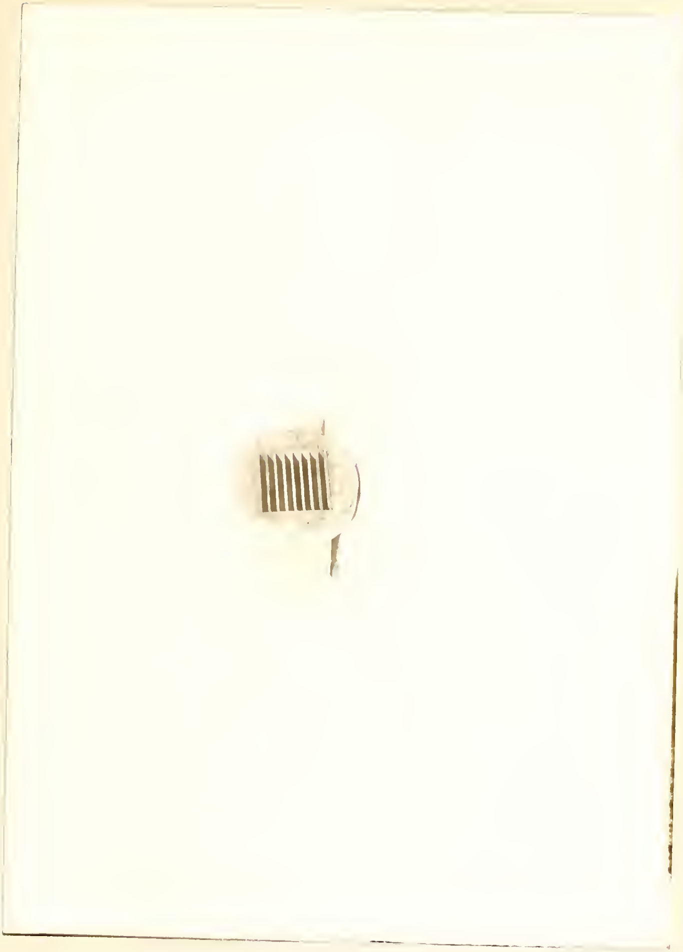
FIGURE 6.



PHOTOGRAPH OF A 3" HYPERBOLIC ELBOW
LOOKING AT ELBOW FLANGE.

36. Figures 5 and 6 are photographs of the three inch hyperbolic elbow used in this investigation. In Fig. 5 are shown the projecting fittings which are used for connecting the inner and outer cores of the elbow to the two sides of a manometer; it will be noted that nothing is inserted or projected into the stream of water, proper, all disturbances which may unduly influence the plane stream line flow being avoided. Fig. 5 illustrates the rectangularity of the vertical cross section inside the elbow. Fig. 7 is a photograph of the grids and the housing for the grids. As shown in Fig.3, the grids and housing are fitted into a special flange which replaces the original flange, facing the flow.

FIGURE 7



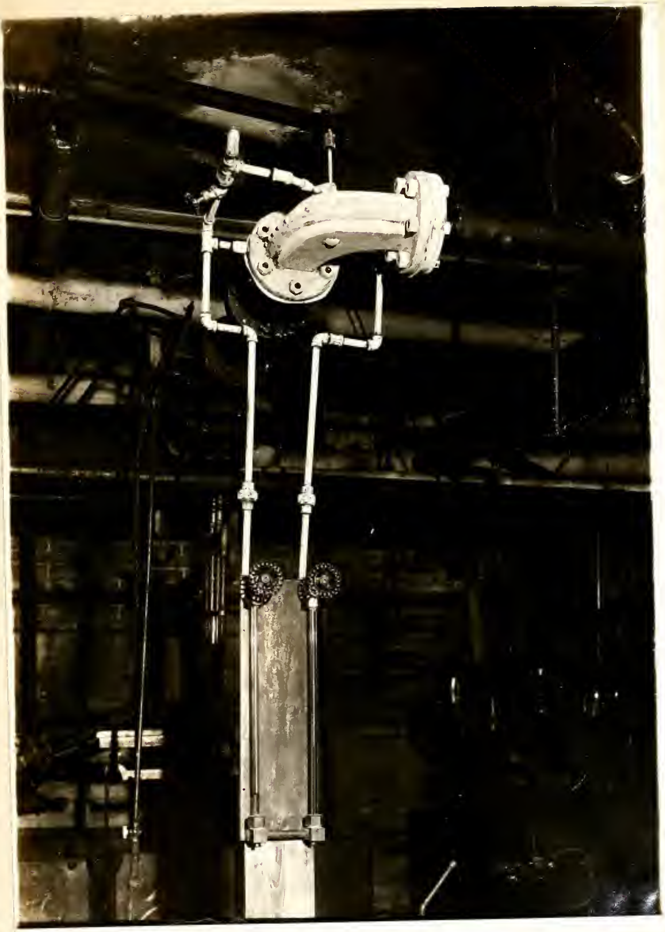
PHOTOGAPH OF GRIDS AND GRIDS HOUSING

A 3" HYPERBOLIC ELBOW

38. The manometer used in these experiments is shown connected to the hyperbolic elbow in Fig. 8. The fittings and valve shown in the upper left hand corner of the photograph connects the top core along the line of symmetry of the elbow with the left side of the manometer. Thus, if the pressure at any point along the line of symmetry, such as 31,32,33,34, etc., (see Fig. 3) was to be compared with the pressure at the center of the inner hyperbola, viz., at 26, then, (1) the extension screws were taken out from point 26 as well as from the particular point on the top of the elbow, say 31, for which pressure comparisons were to be made. All other points were kept plugged up by keeping their extension screws in place tight. (2) The globe valve previously mentioned and shown in the upper left hand corner of Fig. 8 was opened. (3) Finally, the two manometer valves were opened. The pressure on the left side of the manometer corresponded then to that of point 31, while the pressure on the right side of the manometer corresponded to that at point 26.

40. If the pressure at any point on the outer curve was to be compared with that on the inner curve, the extension screws from the two desired points were

FIGURE 8.



PHOTOGRAPH
OF

MANOMETER CONNECTED TO THE

HYPERBOLIC ELBOW

taken out, and the globe valve in the upper left hand corner was closed. As a result, the pressure of the outer core instead of that in the top core was transmitted to the left side of the manometer. In this investigation, only point 6 (See Fig. 3) of the outer curve was compared with point 26 of the inner curve. All pressures along the line of symmetry were compared with the pressure at point 26 as a reference.

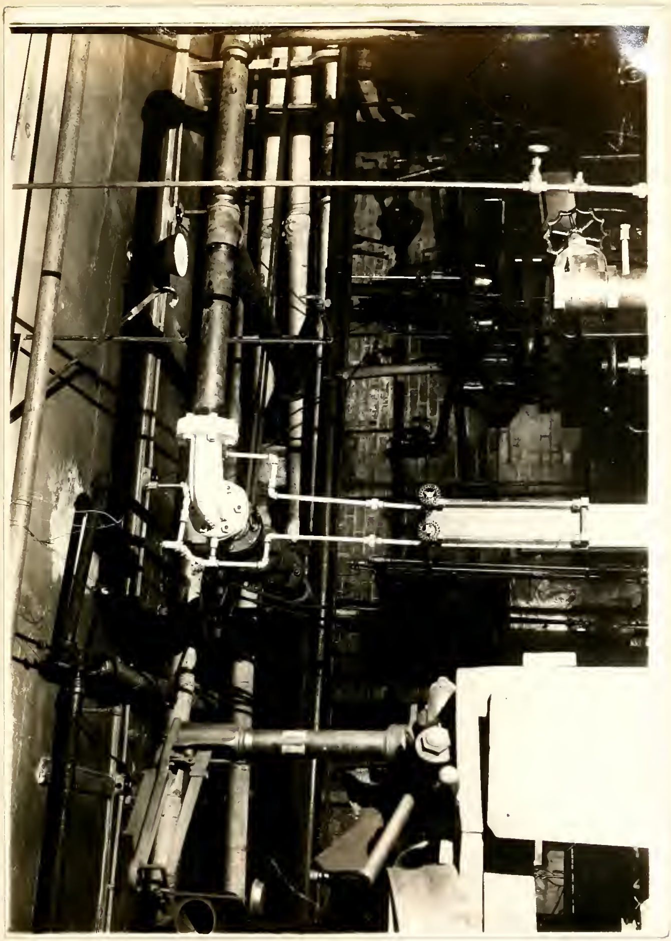
41. To magnify the manometer readings, while testing the hyperbolic elbow for flow of water, a special oil, heavier than water, (S.P.=1.25) was used instead of mercury. The sensitivity was thereby increased $\frac{13.6 - 1}{1.25 - 1} = \underline{50.4}$ times. Since the length of the manometer was about 15", it was possible to use this unusual sensitivity only for moderate velocities of water up to about 3'/sec. For higher velocities, this oil was replaced by acetylene tetrachloride, (S.P.=1.59), whose sensitivity was only $\frac{13.6-1}{1.59-1} = \underline{21.4}$ times that of mercury.

42. Fig. 9 shows a photograph of the manometer, the elbow and a large portion of the piping through which the water was flowing. An inspection of the numerous fittings such as ells, tees, flanges, reducer from 2 1/2" to 3", valves etc., as well as

of the unusual number of 90° turns (7 of them) in such a comparatively small space shows that for producing complex flow the nature of the piping in this case was most favorable. Still the hyperbolic elbow with its straightening grids braved to take a position right in the midst of the maze of fittings and piping with results, which, as will appear later, are, to say the least, gratifying.

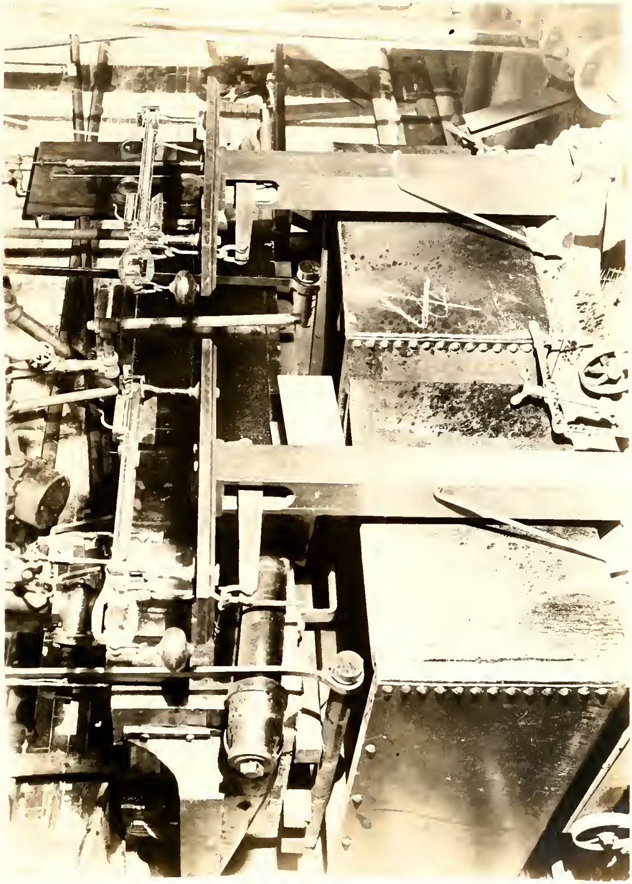
43. The longest pipe in Fig 9. extending from the right flange of the elbow way to the extreme right of the picture, has a tee at about its middle. This tee connects with another 3" pipe leading off, at right angles to the first pipe, to an ell. Another pipe leads down through a valve into a weir (see Fig 10), the latter overflowing into either one of two weighing tanks, shown in Fig. 10. Between the two weighing tanks is pivoted a reversing pan, the bottom of which is v -shaped, with the point of the v at the pivot, and the sides of the v making an angle of about 120° with each other. Because of this reversing pan, it was possible to open the valve until a desired manometer reading was secured, the water being allowed to flow into one of the weighing tanks, whose discharge gate was left open.

FIGURE (9)



PHOTOGRAPH
OF
MANOMETER, ELBOW AND PIPING
FOR
MEASURING FLOW OF WATER.

FIGURE 10.



PHOTOGRAPH OF WEIGHING TANKS.

Meanwhile, the second tank was being weighed empty, and as soon as the flow was decided to be satisfactory enough, the gate of the second tank was closed, the pan was reversed, and the stop watch was started going, the water beginning to fill the weighed tank. When the flow happened to be so rapid that the second weighing tank was filled in too short a period of time, then after the first tank was weighed and its gate closed, the pan was reversed again, causing the water to flow into the first tank; meanwhile, the second tank was weighed while full, then emptied, an after the desired length of time passed, the pan was again reversed into the second tank, while the first was being weighed. The sum of the two net weights then corresponded to the total discharge for the period of time chosen.

44. In the case of steam, the weighing was done after the steam was first condensed in a surface condenser and then emptied into a weighing tank, whose original weight was predetermined.

45. An inclined mercury manometer in addition to the manometer previously mentioned was used during the steam tests. This manometer had an inclination of 10° and was about 15" long. It has not

been finished until after the water tests were completed, and was, therefore, used for the steam tests alone. An 1/8" bore inclined glass tube was used for the manometer, the other side consisting of a 2" nipple. The ratio of areas was, therefore,

$(2:1/8)^2 = 16^2 = 256$, or the readings of the inclined manometer would have been multiplied by

$1 + 1/256 = 1.0039$ to find the actual difference of pressure. The correction is only between 1/3% and 1/2% and may for most practical purposes be neglected. The manometer could be read to 1/16" along the inclined scale which corresponded to $1/16 \times .173 = .0108$" or about 1/100" of mercury head, since sine $10^\circ = .173$.

46. For the flow of steam, in addition to the apparatus described, a pressure gauge and a throttling calorimeter were used.

DETAILS OF TEST

47. In Fig.3 can be seen the spacing of the hyperbolic elbow openings and the system of their numbering. As mentioned previously this investigation was limited to pressure distribution along the line of symmetry, and pressure readings were taken only at the following points:

6, 31, 32, 33, 34, 26,

All pressures were compared with that at point 26. Because of certain doubts regarding the exact location of point 35, the latter point was not investigated.

48. For [redacted] flow measurement, the difference of pressure between points 26 & 6 is most commonly taken. For this reason, special attention was paid to comparing the pressure at point 6 with that at point 26, the number of readings taken for this combination being far in excess of that taken for other points.

49. In starting any run for measuring flow, particular attention was paid to insure the absence of any air bubbles in the manometer or its fittings. To accomplish this result, the city water or high pressure water from a turbine, as the case may have been, was turned on while the valve at the weir was kept open. After the surging action due to air in the piping, was not noticeable any more, and the flow was comparatively steady, the valve at the weir was closed leaving the piping filled with water. Then the two plugs on the top and the two plugs on the bottom of the manometer were removed, and the manometer valves were opened. (see Fig. 8)

The water in the piping now began to flow through the manometer fittings and the manometer glass tubes, finally escaping through the bottom openings of the manometer. Since the water supply valve was still open, any water escaping from the piping was immediately replaced, leaving no chance for air to get entrained in the piping. After the manometer fittings and the manometer glass tubes were sufficiently flushed, the two valves of the manometer were closed, the water in the manometer glass tubes being allowed to escape through the bottom openings. The latter were then plugged up, and the special oil or the acetylene tetrachloride, as the case might have been, was then poured through the two upper openings of the manometer until about the middle of the scale was reached. Finally, water was poured in on top of the oil into both openings, until the water began to overflow from the latter, care having been taken that equal amounts were poured into both sides as evidenced by similar oil levels in the two manometer tubes. The upper plugs were then screwed in, and the manometer valves were opened, thus connecting the manometer through its fittings with the hyperbolic elbow and the water line. That there was no air entrained anywhere in the system was evidenced by

the zero manometer readings not being changed when the manometer valves were opened.

50. The valve at the weir was next opened gradually until the desired manometer reading was obtained, the water being allowed to escape through one of the weighing tanks into the sump. The second tank was now weighed while empty, and the reversing pan was then tipped toward this tank, the stop watch being started simultaneously with the beginning of the filling of the second tank.

51. After a two or three minute run, the pan was reversed into the first tank, the second tank was weighed with its water in, and the weights of the empty and filled tanks as well as the time of the run were entered on the data sheet. Similar entries were made for the manometer readings during the run.

52. The valve over the weir was then further opened until the desired increase of the manometer reading was secured and the experiment was repeated. Over 40 such readings were taken by the writer just for openings 26-6 alone, and over 60 additional readings for 26-6 were taken in an official test under the supervision of Prof. Gebhardt, of Armour Institute of Technology. Also, about 160

readings were taken by the writer for openings 26-31,26-32, 26-33 and 26-34 for water alone.

53. When the readings for 26-6 were completed, a change to openings 26-31 was made, as follows:

The manometer valves were closed. Then the water supply valve was closed, and the water in the piping was allowed to empty through the discharge valve into the weir. The manometer fittings, however, still remained filled with water. Next, the screw opposite opening _6_ was replaced by an extension screw, thus, effectively closing out opening _6_ and disconnecting it from the left side of the manometer. Instead, the extension screw in opening _31_ was replaced by an ordinary screw, and by opening the valve shown in the upper left hand corner of the manometer. Then, the water supply valve was opened again, and, when the flow became steady, the piping was again effectively freed from all entrained air. Finally the discharge valve was closed, and the manometer valves were opened, the zero manometer readings being noted. The apparatus was then ready for another set of readings for 26-31.

54. After completing the readings for 26-31, changes in connections were again made in the manner just described to openings 26-32, 26-33 and 26-34, respectively, a complete set of readings having been taken for each of these connections in order named.

55. In the case of steam, the inclined manometer was used, but in other respects, the test was similar to that for water, except that, for steam, additional readings were taken of the pressure of the steam and of the temperature of the throttling calorimeter for determing the density and quality of the steam whose flow was to be measured.

RESULTS AND CALCULATIONS

56. The data obtained for the flow of water and steam through a 3" hyperbolic elbow are given below. In order to illustrate the method of the calculations, a sample set of results and calculations for openings 26-6 for water and for steam is also included. This refers to calculating the numerical value of the co-efficient k of the formula.

$$v_a = k \sqrt{2gH} \qquad \text{(See eq.16 and eq.30)}$$

In addition, the data for pressure distribution along the line of symmetry for both water and steam is given below, and the determination of the F curves for the two respective cases is illustrated.

57. To begin with, the values of and relations between different fixed items of this test are tabulated below, under the title indicated.

GENERAL FLOW DATA

Manometer Data

Oil, S.P. = 1.25; Acetyl. Tetrach, S.P. = 1.59

$$1'' \text{ Water} = \frac{62.5}{12} \; \# \;/\text{sq. ft.}$$

$$1'' \text{ Oil} = \frac{62.5}{12}(1.25-1) = \frac{62.5}{12} \times .25 \; \# \;/\text{ sq.ft.}$$

If $p = \#$ / sq.ft. and $h_o =$ in. of oil, then, $p =$

$= \frac{62.5}{12} \times .25\, h_o = 1.3\, h_o$ # /sq. ft.

Similarly for h_a = in. of acetylene tetrach., we get, $p = \frac{62.5}{12} \times (1.59-1) h_a = 3.07 h_a$ #/sq. ft.

For the vertical mercury manometer, we get for h_m = in. of mercury.

$p = \frac{62.5}{12} \times (13.6-1) h_m = \underline{65.6\, h_m}$ #/sq. ft.

For the inclined mercury manometer, we get for l_m = in. of mercury along inclination.

$$p = \frac{62.5}{12} \times (13.6-1) h_m =$$

$$= 65.6\, h_m = 65.6\, l_m \sin 10^\circ =$$

$$= 65.5 \times .173\, l_m = 11.4\, l_m$$

DIMENSIONS

Pipe Diameter - 3.067"

Hyperbolic Elbow

(Section through plane of symmetry) -

$3\frac{1}{2}$"x2 1/8"

For location of openings, see Fig.3.

Areas

Pipe , $a = .785 \left(\frac{3.067}{12}\right)^2 = .0514$ sq.ft.

Elbow , $a_1 = \frac{3\frac{1}{2} \times 2\frac{1}{8}}{144} = .0516$ sq.ft.

Ratio , $\frac{a_1}{a} = 1$, practically.

WATER

VELOCITIES

Pipe $v = \frac{W}{62.5 \times 60a} = \frac{W}{62.5 \times 60 \times .0514} =$

$= .00519W$ ft./sec.

(W = Lbs. /min)

Elbow , $v_1 = K\sqrt{2gH} =$

$$= \frac{W}{62.5 \times 60 a_1} = \frac{W}{62.5 \times 60 \times .0516} =$$

$= .00516\ W$ ft/sec.

DISCHARGE

Pipe , $\underline{W}$ is measured; Lbs/min

Oil , $W = 60\ w\ a_1\ v_1 =$

$$= 60 \times 62.5 \times .0516 k\sqrt{2gH} =$$

$$= 60 \times 62.5 \times .0516 k\sqrt{2g\frac{p}{D}} =$$

$$= 60 \times 62.5 \times .0516 K\sqrt{\frac{2g \times 1.3 h_o}{62.5}} =$$

$$= 224\ k\sqrt{h_o}\ \text{, lbs/min}$$

Acetylene
Tetrach, $W = 60\ wa_1\ v_1 =$

$$= 60 \times 62.5 \times .0516\ K\sqrt{2g\frac{3.07}{62.5}h_a}$$

ELBOW $= 344\ k\sqrt{h_a}$ lbs./min

Mercury, in a vertical manometer,

$$W = 60\ wa_1V_1 =$$

$$= 60 \times 62.5 \times .0516\ K\sqrt{2g\frac{65.6\ h_m}{62.5}} =$$

$$= 1590\ k\sqrt{h_m}\ \text{Lbs/min}$$

Mercury in an inclined manometer,

$$W = 60\ wa_1V_1 =$$

$$= 60 \times 62.5 \times .0516 k\sqrt{2g\frac{11.4\ l_m}{62.5}}$$

$$= 662.5\ k\sqrt{l_m}\ \text{, lbs./min}$$

$p = HD$; therefore, $H = \frac{p}{D}$

Coefficients. (Referred to mean velocity through plane of symmetry of elbow).

$$\text{Oil} \;,\; K = \frac{W}{224\sqrt{h_o}}$$

$$\text{Acetyl. Tetrachl.}, \; K = \frac{W}{344\sqrt{h_a}}$$

Mercury, vertical manometer,

$$K = \frac{W}{1590\sqrt{h_m}}$$

Mercury, inclined manometer,

$$K \qquad \frac{W}{662.5\sqrt{l_m}}$$

STEAM

Velocities

$$\text{Pipe,} \quad v = \frac{W}{60\, D_1\, a_1} = \frac{W}{60 \times .0514\, D_1} =$$

$$= .324 \frac{W}{D_1} \quad \text{ft/sec.}$$

D_1 = Density of the steam lbs./cu.ft.

ELBOW,

$$v_1 = K\sqrt{2\,gH} =$$

$$= \frac{W}{60 D_1 a_1} = \frac{W}{60 \times .0516\, D_1} =$$

$$= .323 \frac{W}{D_1} \quad \text{ft/sec.}$$

$$60\, v_1 = 19.4 \frac{W}{D_1} \quad \text{ft/min.}$$

DISCHARGE

DISCHARGE

Oil , $$W = 60\ D_1 a_1 v_1 =$$

$$= 60\ D_1 \times .0516\ k\sqrt{2g \frac{p}{D_1}} =$$

$$= 60 \times .0516\ k\sqrt{2\ g\ p\ D_1} =$$

$$= 60 \times .0516\ k\sqrt{2g \times 1.3 h_o D_1} =$$

$$= \underline{28.3\ k\sqrt{h_o D_1}}\ \text{lbs./min}$$

Acetyl.Tetrachl

$$W = 60\ D_1 a_1 v_1 =$$

Elbow $$= 60 \times .0516\ K\sqrt{2g \times 3.07 h_a D_1}$$

$$= \underline{43.6\ K\sqrt{h_a D_1}\ \text{lbs. min.}}$$

Mercury in a vertical manometer,

$$W = 60\ D_1\ a_1\ v_1 =$$

$$= 60 \times .0516\ K\sqrt{2g \times 65.6 h_m D_1} =$$

$$= \underline{201.5\ K\sqrt{h_m D_1}\ \text{Lbs.min.}}$$

Mercury in an inclined manometer.

$$W = 60\ D_1\ a_1\ v_1 =$$

$$= 60 \times .0516\ K\sqrt{2g \times 11.4\ l_m D_1} =$$

$$= 84\ K\sqrt{l_m D_1}\ \text{Lbs./min.}$$

Coefficients

Oil , $$k = \frac{W}{28.3\sqrt{h_o D_1}}$$

Acetyl. Tetrachl., $K = \dfrac{W}{43.6\sqrt{h_a D_1}}$

Mercury, vertical manometer,

$$K = \frac{W}{201.5\sqrt{h_m D_1}}$$

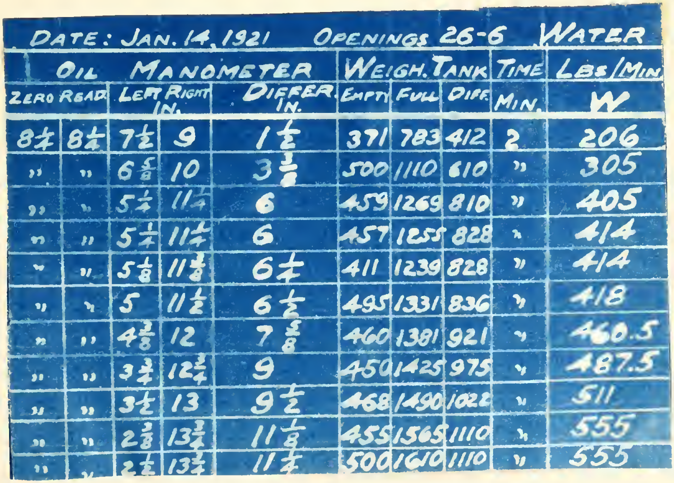
Mercury, inclined manometer,

$$K = \frac{W}{84\sqrt{l_m D_1}}$$

Coefficients for Water

58. Below, in Table 1, is given a sample of date taken for water with openings 26-6. As will be noted, the manometer

TABLE 1



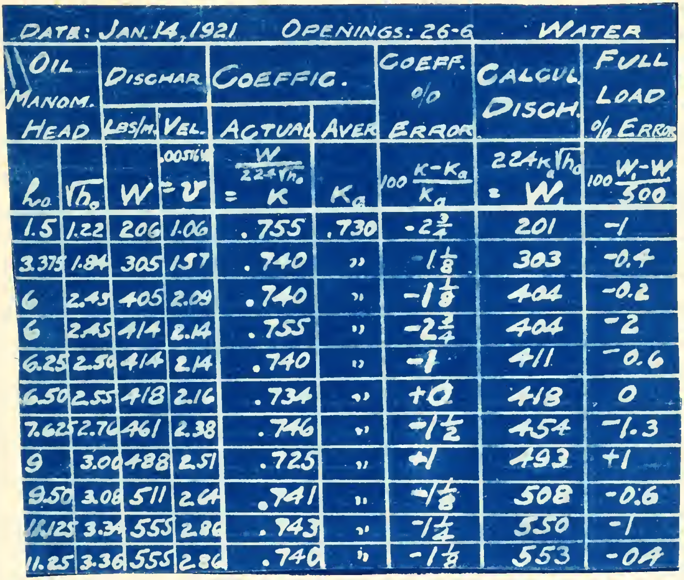
DATE: JAN. 14, 1921 OPENINGS 26-6 WATER

OIL MANOMETER					WEIGH. TANK			TIME	LBS/MIN
ZERO	READ.	LEFT	RIGHT IN.	DIFFER. IN.	EMPTY	FULL	DIFF.	MIN.	W
8 1/4	8 1/4	7 1/2	9	1 1/2	371	783	412	2	206
"	"	6 5/8	10	3 3/8	500	1110	610	"	305
"	"	5 1/4	11 1/4	6	459	1269	810	"	405
"	"	5 1/4	11 1/4	6	457	1255	828	"	414
"	"	5 1/8	11 3/8	6 1/4	411	1239	828	"	414
"	"	5	11 1/2	6 1/2	495	1331	836	"	418
"	"	4 3/8	12	7 5/8	460	1381	921	"	460.5
"	"	3 3/4	12 3/4	9	450	1425	975	"	487.5
"	"	3 1/2	13	9 1/2	468	1490	1022	"	511
"	"	2 3/8	13 3/4	11 1/8	455	1565	1110	"	555
"	"	2 1/2	13 3/4	11 1/4	500	1610	1110	"	555

zero readings did not change throughout this set of runs. Two minute runs were taken, and the discharge for the different runs has been varied from about 200 to about 550 lbs. of water per minute.

59. In Table 2, are given the calculations of velocities and coefficients for the set of data found in Table 1.

TABLE 2



DATE: JAN. 14, 1921 OPENINGS: 26-6 WATER

OIL MANOM. HEAD		DISCHAR.		COEFFIC.		COEFF. % ERROR	CALCUL. DISCH.	FULL LOAD % ERROR
		LBS/M.	VEL.	ACTUAL	AVER.			
h_o	$\sqrt{h_o}$	W	$.00516W = v$	$\frac{W}{224\sqrt{h_o}} = K$	K_a	$100\frac{K-K_a}{K_a}$	$224K_a\sqrt{h_o} = W_1$	$100\frac{W_1-W}{500}$
1.5	1.22	206	1.06	.755	.730	$-2\frac{3}{4}$	201	-1
3.375	1.84	305	157	.740	"	$-1\frac{1}{8}$	303	-0.4
6	2.45	405	2.09	.740	"	$-1\frac{1}{8}$	404	-0.2
6	2.45	414	2.14	.755	"	$-2\frac{3}{4}$	404	-2
6.25	2.50	414	2.14	.740	"	-1	411	-0.6
6.50	2.55	418	2.16	.734	"	+0	418	0
7.625	2.76	461	2.38	.746	"	$-1\frac{1}{2}$	454	-1.3
9	3.00	488	2.51	.725	"	+1	493	+1
9.50	3.08	511	2.64	.741	"	$-1\frac{1}{8}$	508	-0.6
11.125	3.34	555	2.86	.743	"	$-1\frac{1}{2}$	550	-1
11.25	3.36	555	2.86	.740	"	$-1\frac{1}{8}$	553	-0.4

SAMPLE CALCULATIONS OF

VELOCITIES AND CORRESPONDING

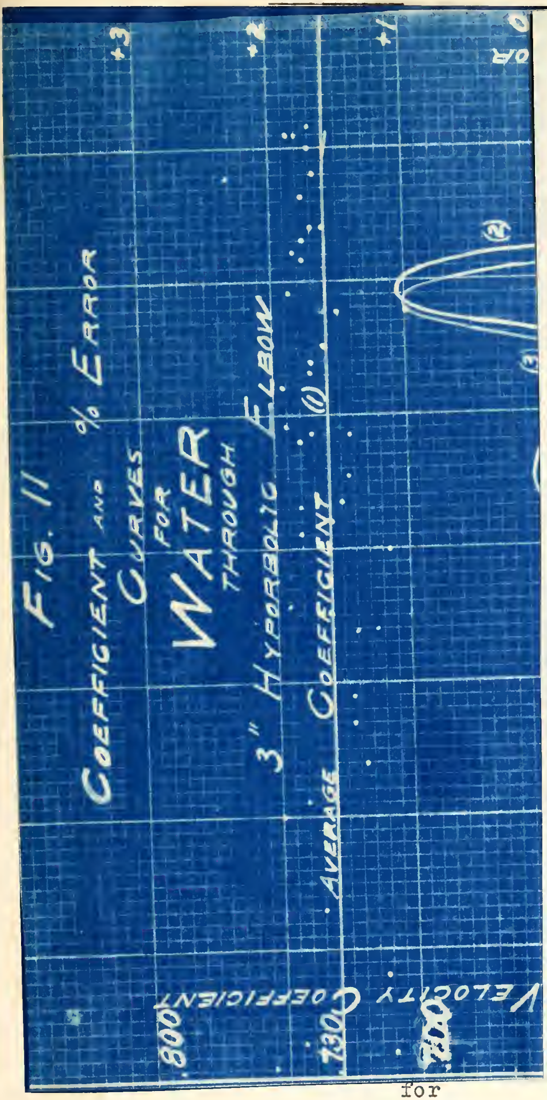
COEFFICIENTS
FOR
WATER

By average coefficient is meant the final average as determined from the complete set of data for connection 26-6. By full load o/o error is meant the error of calculated discharge when compared with full load discharge, the latter being taken for this case as 500 lbs./min.

60. The values of the coefficient of connection 26-6 as calculated from all the data taken for water flowing through the 3" hyperbolic test elbow are plotted in Fig. 11, as ordinates with discharges in lbs/min. as absicissae. The fact that such comparatively small differences exist between different values of the coefficient and the average coefficient is striking. The errors for the coefficients of table 2 have a maximum error of less than 3% at two points as shown by the coefficients % error curve marked 2, the majority of the coefficients having errors of less than 2%. If the errors are figured on a basis of the difference between calculated flow and actual flow as compared with full load flow, these errors are still further diminished as illustrated by curve 3, marked "full load % error curve". The maximum error in this case is only 2%, most of the errors being less than 1%.

61. In general, it has been found from all

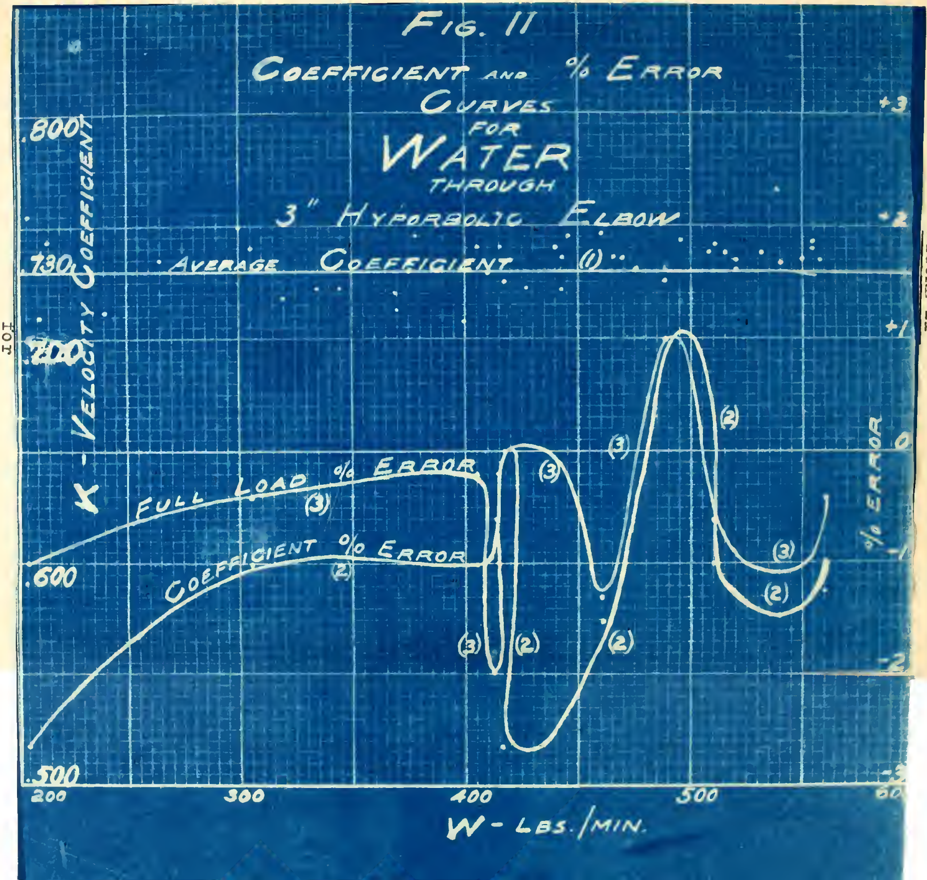
FIGURE 11



for

WATER THROUGH 3" HYBERBOLIC ELBOW.

FIGURE 11



WATER THROUGH 3" HYPERBOLIC ELBOW.

the data taken that about 90% of all the coefficients of connection 26-6, which is the one intended for flow measurement, were accurate within 97.5% or better. The maximum error was found to be only 4%. If these errors were figured on a basis of % full load, they would be still further reduced in the same manner as they were for the errors calculated for Table 2 and plotted as curve 3 of Fig 11.

62. It may be in order to mention at this point that the final analysis of the pressure distribution curve, given further down, seems to indicate that for flow of water through the hyperbolic elbow the effect of the friction is almost negligible, at least for velocities up to about 3'/sec. This may be illustrated for connection 26-6 by comparing the actual coefficient just found with the theoretical one for frictionless flow. Thus, for eq. (18) we have,

$$K = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}}$$

Now for our 3" elbow,

R = 5.5"
R = 2 "

Therefore,

$$K-\sqrt{\frac{5.5+2}{4(5.5-2)}} = \sqrt{\frac{7.5}{4 \times 3.5}} = \sqrt{.535} =$$

$$= \underline{.732} \qquad (31)$$

But this is almost exactly the same value as that of the coefficient found experimentally. This fact illustrates the contention made above, that the effect of friction upon the actual flow through the hyperbolic elbow is comparatively small.

63. Furthermore, it would be expected that if the motion of water around a hyperbolic elbow actually approached a plane and stream line flow, the differences of pressure between any two points along the line of symmetry ought to all follow the same law. The exact nature of this law will be deduced later from the pressure distribution curve, but the approximate values of the coefficients ought to be the same as those for frictionless flow, since this fact was found to be true for connection 26-6.

64. Now, the general formula for the coefficient K of equation

$$v = K\sqrt{2gH}$$

was mentioned to be of the form,

$$K = \sqrt{\frac{(R_2 + R_1)^2}{4(r_2^2 - r_1^2)}} \qquad \text{(see eq. \quad)}$$

this taking the special form of

$$K = \sqrt{\frac{R_2 + R_1}{4(R_2 - R_1)}}$$

for $r_1 = R_1$ and $r_2 = R_2$, i.e. for connection 26-6. This formula gives the following theoretical values for the coefficients of connections 26-31 26-32, 26-33, and 26-34, for frictionless flow, given in TABLE 3.

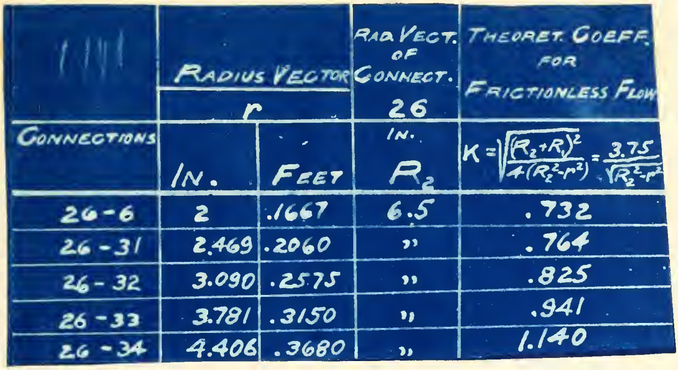
65. A comparison of the theoretical values of the coefficients for the various connection as given in Table 3 with the actual values of these coefficients as found experimentally for water and given in Table 4, indicates that our original deduction that the effect of friction on the flow of water through our hyperbolic elbow is small, is verified by all the results of our test. Since only the approximate existence of a stream line flow, along rectangular hyperbolas, could give us such close agreement between theoretical and actual coefficients for the different connections along the line of symmetry, is it not permissible to assume that the desired plane stream lone motion was actually secured by means of the straightening grids and the hyperbolic elbow?

66. With such an assumption, it ought to be possible to calculate the value of the coefficient K of the flow measurement formula,

$$v_a = K\sqrt{2\,gH}$$

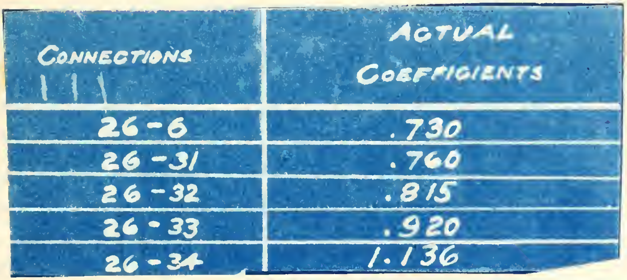
for any hyperbolic elbow from its dimensions and

TABLE 3



Connections	Radius Vector r		Rad. Vect. of Connect. 26 In.	Theoret. Coeff. for Frictionless Flow
	In.	Feet	R_2	$K = \sqrt{\frac{(R_2+R_1)^2}{4(R_2^2-r^2)}} = \frac{3.75}{\sqrt{R_2^2-r^2}}$
26-6	2	.1667	6.5	.732
26-31	2.469	.2060	"	.764
26-32	3.090	.2575	"	.825
26-33	3.781	.3150	"	.941
26-34	4.406	.3680	"	1.140

TABLE 4



Connections	Actual Coefficients
26-6	.730
26-31	.760
26-32	.815
26-33	.920
26-34	1.136

ACTUAL COEFFICIENTS FOR WATER

from the location of the points of connection. It does not seem to make any difference how complicated the flow may be or what the nature of the piping and fittings is, the straightening grids and the centrifugal force seem to very nearly destroy the complexity of the original flow, changing it to plane stream line flow. Therefore, the hyperbolic elbow does not need any local calibrations, its coefficient being of a fixed, reliable and known value based upon the elbow design and upon the points of connections only. It is hoped that future tests will confirm the results and conclusions of this investigation of flow of water through rectangular hyperbolic elbows.

PRESSURE DISTRIBUTION FOR WATER

66. It will be remembered that each pressure connection such as 26-6, 26-31 etc., was investigated separately so that the discharges measured for the different pressure connections were not exactly identical although they were not very far apart. Therefore, if we wish to analyze the pressure distribution along the line of symmetry for any chosen discharge, say 500 lbs/min., it is evident that, if for one connection the

nearest date taken is 490 lbs/min, while for another connection this happens to be 510 lbs/min, then, in the first case, the pressure difference is not large enough for 500 lbs/min, while, in the second case, the pressure difference is too large; in other words, the pressure readings at the different connections must be reduced to a similar discharge basis.

68. Again, it will be remembered that the average velocity, v_a, is proportional to the square root of the corresponding difference of pressure, p; viz.,

$$v_a = K\sqrt{2g\,H} = K\sqrt{2g\,\frac{p}{D}} \qquad \text{(see 30)}$$

Also, the rate of discharge W, is proportional to the average velocity, v_a ; viz.,

$$W = 60\,D\,a\,v_a = 60 \times .0516\ D\,v_a, \text{Lbs/min} \qquad (31)$$

or

$$W = 3.10\ D\ K\sqrt{2g\,\frac{p}{d}} =$$
$$= 3.10\ K\sqrt{2g\,p\,D} \qquad (33)$$

from which,

$$p = \frac{W^2}{(3.10k)^2\ 2g\,D} =$$
$$= C\,\frac{W^2}{D} \qquad (34)$$

where,

C = constant,

69. The last equation shows that pressure difference changes as the square of the rate

of discharge, W, and also decreases directly with the density D, of the fluid whose flow is measured. Therefore, if, in the case of water any connections the pressure difference was taken for say, 490 lbs/min instead of 500#/min, this pressure difference must be increased $\frac{(500)^2}{490}$ times in order to determine its value for 500 lbs. min. In the case of gases and vapors, such as steam, correction must also be made for density D in accordance with eq.33.

70. Another method is to find from data the average coefficient, K for each connection. Then the pressure difference, p, can be found directly from eq.33, by substituting the proper values for K, W, and, D.

71. Using the second method in the case of water, first, the average coefficients for the different pressure connections must be found from the data, given in the appendix. Then, equation 33 may be simplified for the particular case of W = 500 lbs/min of water, whose density is D = 62.5 lbs/cu.ft. to the following form,

$$p = \frac{W^2}{(3.10\ K)^2\ 2gD} =$$

$$= \frac{500^2}{3.10^2 \times 64.4 \times 62.5\ K^2} =$$

$$= \frac{6.46}{K^2} \qquad (35)$$

72. By using the average coefficients for connections 26-6, 26-31, 26-32, 26-33, and 26-34 the following Table 5, for pressure distribution along the line of symmetry, was compiled. The pressures mentioned were really differences of pressurs between the different openings and opening 26. But the only effect this had on the curve was to shift it down or shift the x-axis up without changing the actual nature of the curve.

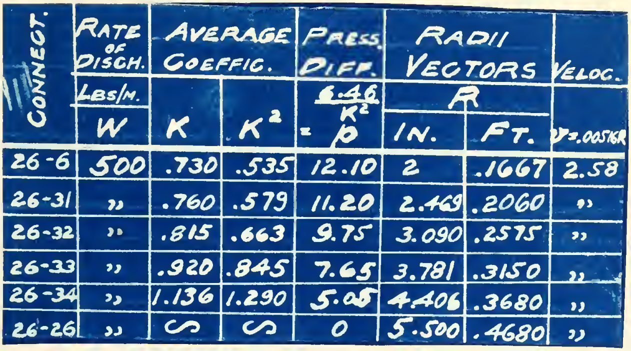
73. The pressure distribution curve, indicating the variation of pressure along the line of symmetry of a 3" long sweep hyperbolic elbow, when water flows through it, is shown in Fig 12. The pressures are relative, inasmuch as they are differences between the pressure at any given connection and that at point 26. Their values and the corresponding radii vectors are those found in Table 5.

"F" CURVE FOR WATER

74. To find the nature of the "F" curve from the pressure distribution curve of Fig. 12, we recollect the relation previously

TABLE 5

WATER



CONNECT.	RATE OF DISCH.	AVERAGE COEFFIC.		PRESS. DIFF.	RADII VECTORS		VELOC.
	LBS/M.			$\frac{6.46}{K^2}$	R		
	W	K	K^2	$= p$	IN.	FT.	$v = .00516R$
26-6	500	.730	.535	12.10	2	.1667	2.58
26-31	"	.760	.579	11.20	2.469	.2060	"
26-32	"	.815	.663	9.75	3.090	.2575	"
26-33	"	.920	.845	7.65	3.781	.3150	"
26-34	"	1.136	1.290	5.05	4.406	.3680	"
26-26	"	∞	∞	0	5.500	.4680	"

PRESSURE DISTRIBUTION
CURVE DATA
FOR

LINE OF SYMMETRY

FIGURE 12.

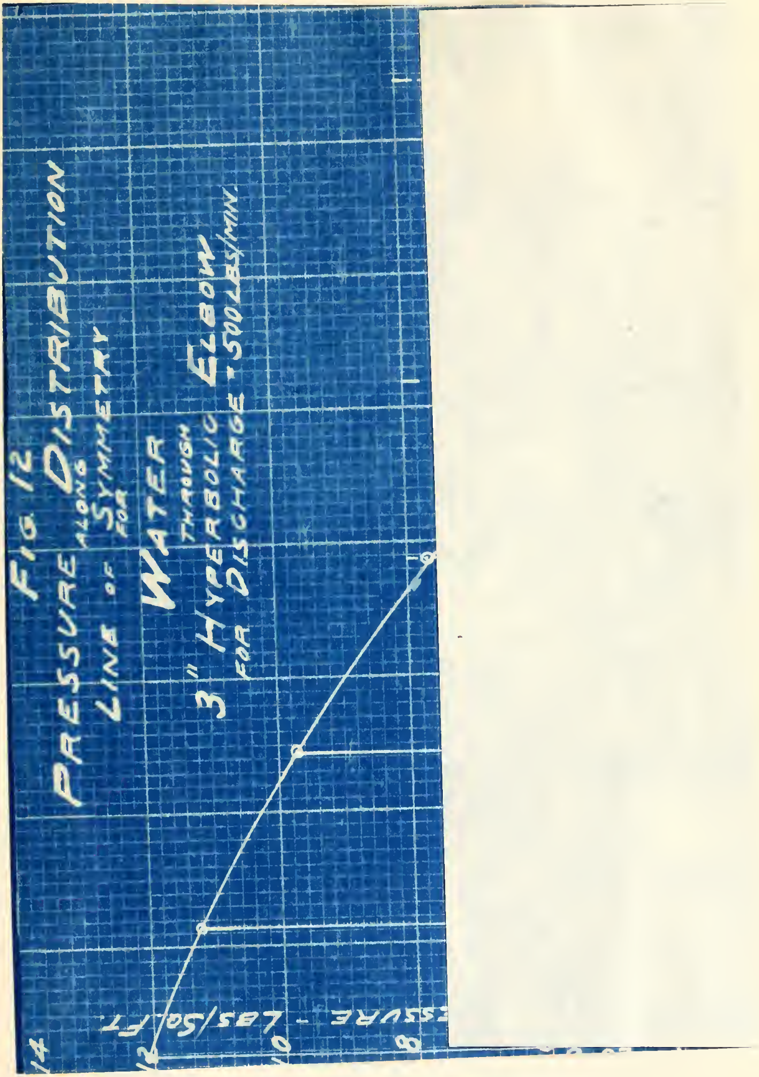
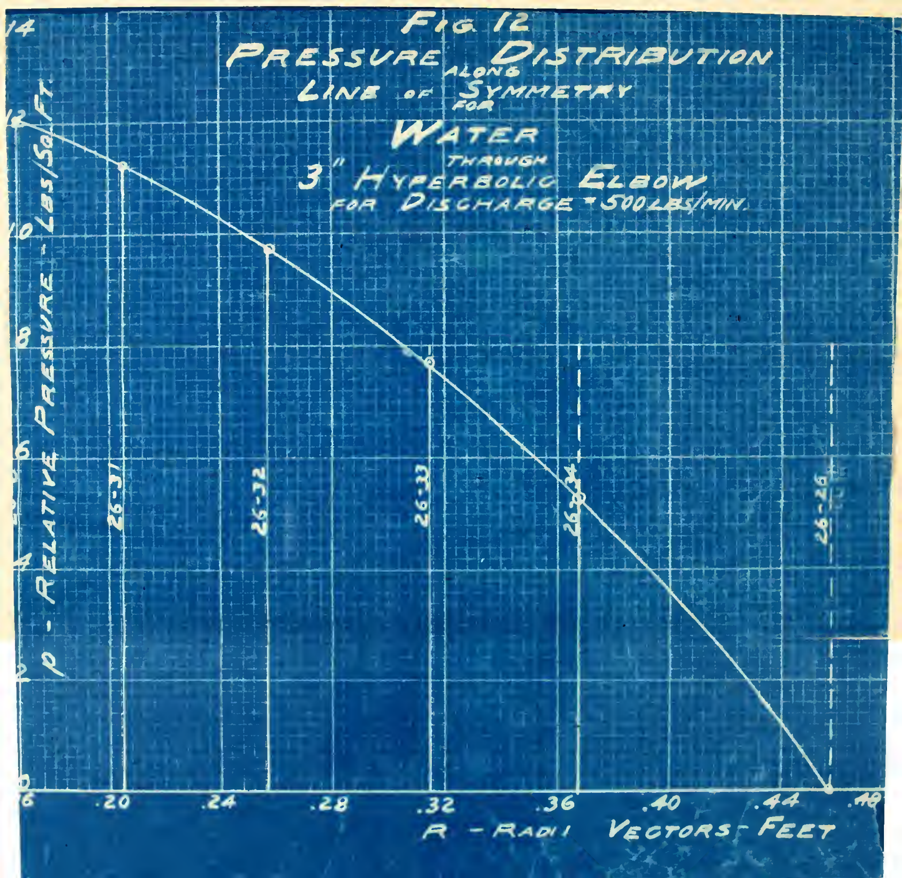


Fig 12
Pressure Distribution
along Line of Symmetry
for
Water
through
3" Hyperbolic Elbow
for Discharge = 500 Lbs/Min.
ESSURE - LBS/SQ FT
12
10
8

FIG. 12

PRESSURE DISTRIBUTION ALONG LINE OF SYMMETRY FOR WATER THROUGH 3" HYPERBOLIC ELBOW FOR DISCHARGE = 500 LBS/MIN.



mentioned viz.,

$$\frac{dp}{dR} = -\frac{D}{g} R F^2 \qquad \text{(See eq. 20)}$$

From this equation, we can express the unknown function "F" in terms of the rate of pressure variation, $\frac{dp}{dR}$, along the line of symmetry, viz.,

$$F(R) = \sqrt{\frac{g}{D}} \sqrt{\frac{-1}{R} \frac{dp}{dR}} \qquad (36)$$

For water $D = 62.5$ lbs/cu.ft., and the last equation is simplified to

$$F(R) = .717 \sqrt{-\frac{1}{R} \frac{dp}{dR}} \qquad (36a)$$

75. Now, from the pressure distribution curve of Fig. 12, for any given value of R, we locate the corresponding point on the curve. Then two points are located at short distances on each side of the point chosen, care being taken that the fraction of the curve between these two points is nearly straight. If, next, the horizontal and vertical projections of this fraction of the curve are measured, their ratio can evidently be expressed as $\frac{\Delta p}{\Delta R}$ and, if the fraction of the curve is straight enough, we have

$$\frac{\Delta p}{\Delta R} = \frac{dp}{dR}, \text{ nearly.}$$

76. To insure accuracy, the pressure distribution curve shown in Fig. 12 was originally

plotted on a much larger scale, and a considerable number of points were investigated for their values of $\frac{dp}{dR}$ in the manner just explained. A table was then made of the values of $\underline{\Delta p}$ $\underline{\Delta R}$ $\frac{\Delta p}{\Delta R}$ and finally of $-\frac{1}{R}\frac{\Delta p}{\Delta R}$ and of $\sqrt{\frac{-1}{R}\frac{\Delta p}{\Delta R}}$

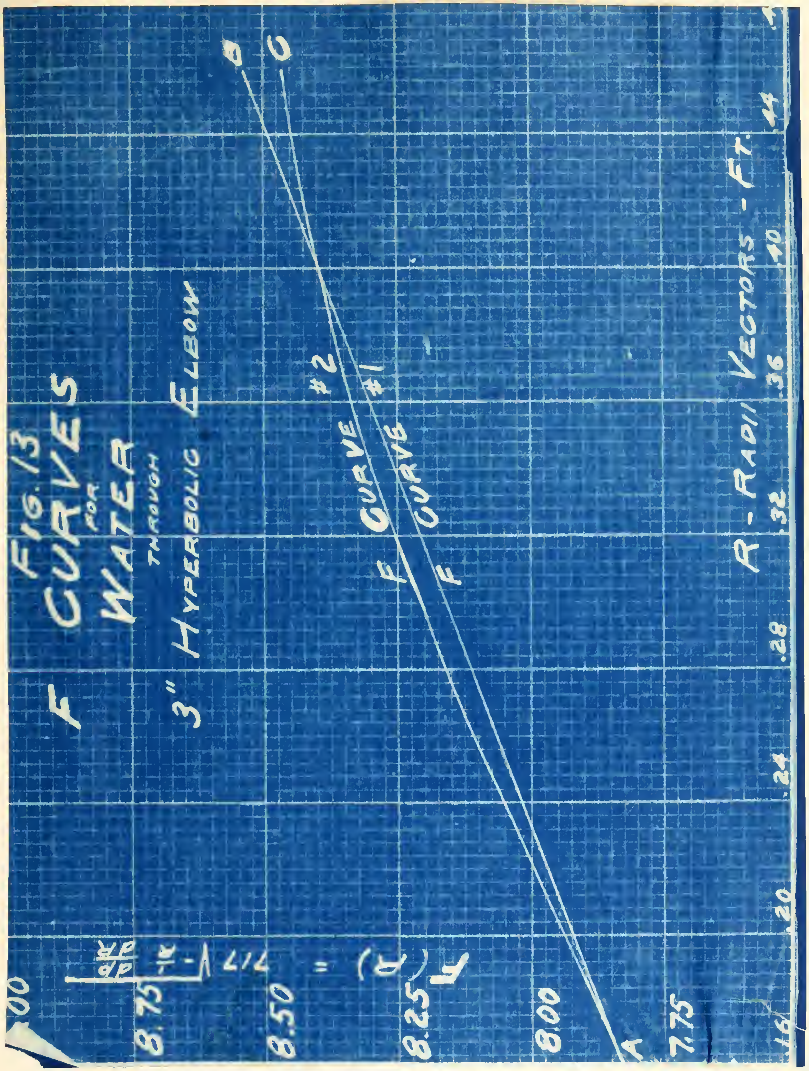
The results of these calculations are given below in Table 6. This table, therefore, gives us the different values of $F(R) = .717\sqrt{\frac{-1}{R}\frac{dp}{dR}}$ for various values of the radii vectors $\underline{R}$ being expressed in feet. The "F" curve can now be plotted from Table 6, with values of F as ordinates , and values of $\underline{R}$ as abscissae. A reproduction of the "F" curve thus found is given in Fig. 13.

77. An inspection of the points plotted for drawing the "F" curve in Fig. 13 shows that, as a first approximation they may be considered to lie on the inclined straight line AB, marked "F" curve #1. More careful analysis however, leads us to assume that the "F" curve closely approaches some form of a higher parabola of the general form.

$$F^n = C^n R \quad (37)$$

This parabola of the $\underline{n}$th degree is represented by the "F" curve #2.

FIGURE 13


F
FIG. 13
CURVES
FOR
WATER
THROUGH
3" HYPERBOLIC ELBOW
$F(R) = 717\sqrt{-R\frac{dP}{dR}}$
F CURVE #2
F CURVE #1
B
C
A
00
8.75
8.50
8.25
8.00
7.75
.16
.20
.24
.28
.32
.36
.40
.44
R - RADII VECTORS - FT.

78. The equation of the "F" curve #1 is found by regular methods, that we need not go into, to be

$$F\ (R) = B(\ 1 + A\ R\ ) =$$

$$= B\ (1 + .348\ R) = 7.425(1 + .348\ R) \quad (38)$$

by modifying the expressions for average velocity pressure etc. with this value of F (R), these expressions take the forms:

$$v_a = B\ \frac{(R + R_1)}{2} + .116\ \frac{(R_2^3 - R_1^3)}{R_2 - R_1} = \quad (39)$$

$$= \underline{.3488\ B} \qquad \text{(for our 3" elbow)}$$

$$p = p_0 - \frac{D}{g}\quad B^2(\ \frac{R^2}{2} + .233\ R^3 + .030\ R^4)$$

$$p_d = p_{26} - p = \frac{D}{g}\ B^2\left[.5\ (R_{26}^2 - R^2) + .233(R_{26}^2 - R^2) + .030(R_{26}^4 - R^4)\right]$$

$$= 107\ \left[\phi(R_{26}) - \phi(R)\right] \qquad (41)$$

and from eq. (41), we get,

$$B = \sqrt{\frac{1}{2\left[\phi\ (R_{26}) - \phi(R)\right]}}\ \sqrt{2g\ \frac{p}{D}} = \sqrt{\frac{1}{2\left[\phi(R_{26}) - \phi(R)\right]}}\ \sqrt{2gH}$$

(42)

while,

$$v_a = B\left[\frac{(R2 + R_1}{2} + .116\ \frac{R_2^3 - R_1^3}{R_2 - R_1}\right] =$$

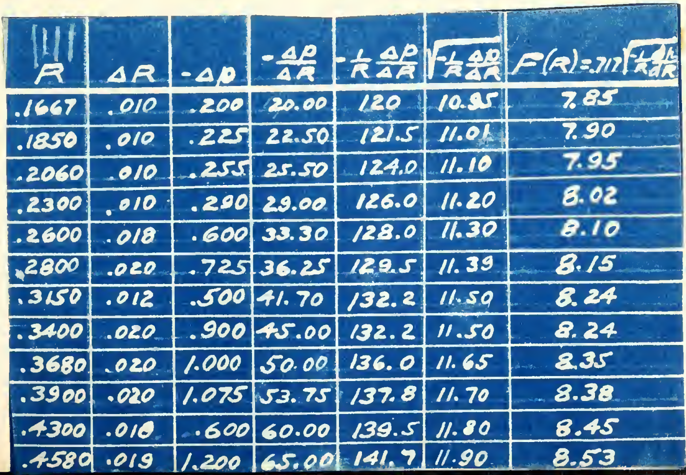
$$= \frac{\left[\frac{R_2 + R_1}{2} + .116\ \frac{R_2^3 - R_1^3}{R_2 - R_1}\right]\sqrt{2\ g\ H}}{\sqrt{2\ \left[\phi(R_{26}) - \phi(R)\right]}} =$$

$$= K\sqrt{2\ g\ H} \qquad (43)$$

and,

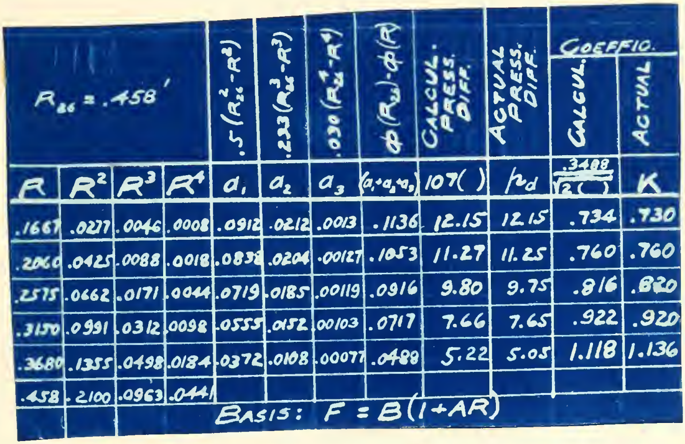
$$K = \frac{\left[\frac{R_2 + R_1}{2} + .116 \frac{(R_2^3 - R_1^3)}{R_2 - R_1}\right]}{\sqrt{2\left[\phi(R_2) - \phi(R)\right]}} = \frac{.3488}{\sqrt{2\left[\phi(R_2) - \phi(R)\right]}} \qquad (44)$$

79. The equations for pressure difference p_d and for the coefficient, K, as given by equations (41) and (44) respectively, make it possible to calculate before-hand the pressure distribution along the line of symmetry as well as the coefficients for the different connections. These calculated values can then be compared with the experimental values, checking the accuracy of finding the "F" curve from the pressure distribution curve. Such a comparison can be made by inspecting Table 7, given below.



R	ΔR	$-\Delta p$	$-\frac{\Delta p}{\Delta R}$	$-\frac{1}{R}\frac{\Delta p}{\Delta R}$	$\sqrt{-\frac{1}{R}\frac{\Delta p}{\Delta R}}$	$F(R) = .717\sqrt{-\frac{1}{R}\frac{dp}{dR}}$
.1667	.010	.200	20.00	120	10.95	7.85
.1850	.010	.225	22.50	121.5	11.01	7.90
.2060	.010	.255	25.50	124.0	11.10	7.95
.2300	.010	.290	29.00	126.0	11.20	8.02
.2600	.018	.600	33.30	128.0	11.30	8.10
.2800	.020	.725	36.25	129.5	11.39	8.15
.3150	.012	.500	41.70	132.2	11.50	8.24
.3400	.020	.900	45.00	132.2	11.50	8.24
.3680	.020	1.000	50.00	136.0	11.65	8.35
.3900	.020	1.075	53.75	137.8	11.70	8.38
.4300	.010	.600	60.00	139.5	11.80	8.45
.4580	.019	1.200	65.00	141.7	11.90	8.53

TABLE 7 WATER



$R_{26} = .458'$				$.5(R_{26}^2 - R^2)$	$.233(R_{26}^3 - R^3)$	$.030(R_{26}^4 - R^4)$	$\phi(R_{26}) - \phi(R)$	CALCUL. PRESS. DIFF.	ACTUAL PRESS. DIFF.	COEFFIC.	
										CALCUL.	ACTUAL
R	R^2	R^3	R^4	a_1	a_2	a_3	$(a_1+a_2+a_3)$	$107(\ )$	p_d	$\frac{.3488}{\sqrt{2(\ )}}$	K
.1667	.0277	.0046	.0008	.0912	.0212	.0013	.1136	12.15	12.15	.734	.730
.2060	.0425	.0088	.0018	.0838	.0204	.00127	.1053	11.27	11.25	.760	.760
.2575	.0662	.0171	.0044	.0719	.0185	.00119	.0916	9.80	9.75	.816	.820
.3150	.0991	.0312	.0098	.0555	.0152	.00103	.0717	7.66	7.65	.922	.920
.3680	.1355	.0498	.0184	.0372	.0108	.00077	.0488	5.22	5.05	1.118	1.136
.458	.2100	.0963	.0441								

BASIS: $F = B(1 + AR)$

COMPARISON BETWEEN

CALCULATED AND ACTUAL

PRESSURES AND CO-EFFICIENTS

BASIS : $F = B(1 + AR)$

The apparent check of calculated and actual values of both, the pressure distribution and of the coefficients, indicates the degree of accuracy of the process of plotting the "F" curve.

80. The equation of the "F" curve #2 has been found by choosing any two values of the F from the curve and substituting them and the corresponding values of R in equation, (37) rewritten for convenience as

$$n \log_{10} F = \log R + n \log C \quad (37)$$

by taking the common log of the two sides of equation (37). Then $n \log C$ was eliminated from the two particular equations for the two chosen points, and the equations were solved for n. This process was repeated and the average of all the values of n thus found was calculated to be about 13. This value of n was next substituted into the original equations from which n was found and a number of different values for $\log C$ were also found. An average value of $\log_{10} C = 0.9494$ was calculated for the latter, corresponding to a value of $C = 8.90$

The final expression for F, therefore, took the form,

$$F = C R^{\frac{1}{n}} =$$

$$= 8.9 R^{1/13} \quad (45)$$

81. When the expression for F corresponding to "F" curve #2 is used, the equation for average velocity, pressures etc. take the following forms:

$$v = R F = CR^{1+\frac{1}{n}} = CR^{14/13} \qquad (46)$$

$$v_a = \frac{n \qquad C}{2n + \quad 1} \frac{(R_2^{27/13} - R_1^{27/13})}{R_2 - R_1} = .[illegible]$$

$$- .484 C \frac{(R_2^{27/13} - R_1^{27/13})}{R_2 - R_1} = \qquad (47)$$

$$= .287\ C$$

$$-\frac{1}{R}\frac{d \quad p}{d \quad R} = \frac{D}{g} F^2 = \frac{D}{g} C^2 R^{2/n} \qquad (48) \text{ or}$$

$$\frac{dp}{dR} = -\frac{D}{g} C^2 R^{\frac{2+n}{n}} \qquad (49) \text{ and}$$

$$p = p_0 - \frac{D}{g} C^2 \left(\frac{n}{2+2n}\right) R^{\frac{2+2n}{2}} = \underline{\qquad\qquad}$$

$$= p_0 - .930 \frac{D}{2g} C^2 R^{28/13} \qquad (50) \text{ while,}$$

$$p_d = p_{26} - p = .930 \frac{D}{2g} C \left(R_{26}^{28/13} - R^{\frac{28}{13}}\right) =$$

$$= 73.5 \quad (.1862 - R^{28/13}) \qquad (51)$$

so that,

$$C = \frac{1}{\sqrt{.930\ (R_{26}^{28/13} - R^{28/13})}} \sqrt{2g\ H} \qquad (52$$

and

$$v_a = .482\ C \frac{R_2^{27/13} - R_1^{27/13}}{R_2 - R_1} =$$

$$= \frac{.482\ (R_2^{27/13} - R_1^{27/13})}{(R_2 - R_1)} \frac{\sqrt{2gH}}{\sqrt{.930\ (R_{26}^{28/13} - R^{28/13})}} \qquad (53)$$

or

$$K = \frac{.482\ \left(R_2^{\frac{27}{13}} - R_1^{\frac{27}{13}}\right)}{(R_2 - R_1)\sqrt{.930\ \left(R_{26}^{\frac{28}{13}} - R^{\frac{26}{13}}\right)}} =$$

$$= \frac{.298}{\sqrt{.1862 - R^{28/13}}} \qquad (54)$$

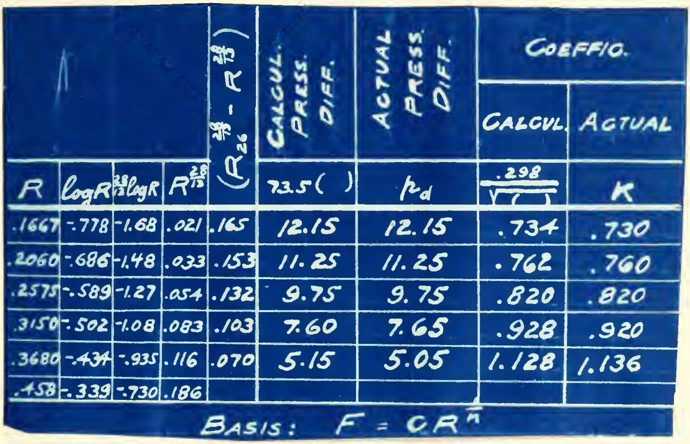
82. The calculations of pressure distribution and of coefficients based upon equations (51) and(54) respectively, are given in Table 8. The agreement between calculated and actual values of pressures and co-efficients is in this case even closer that in the case of "F" curve #1, as it should, since the parabolic form of the "F" curve is more accurate. Furthermore, all the equations are much simplified by the use of "F" curve #2 as evidenced by a comparison of any corresponding equations, e.g. equation (51) and equation (41) for pressure difference. The same is true for all the rest of the equations.

CO-EFFICIENTS FOR STEAM

83. A sample of data taken for steam with opening 26-6 is given below in Table 9, while the calculations of the co-efficients and their accuracies are given in Table 10. It was considered permissible to assume 14.7 lbs/sq.in. absolute pressure inside the calorimeter and to correct density for quality by writing

$$D_1 = \frac{D}{x} \qquad (55)$$

TABLE 8 WATER



				$(R_{26}^{\frac{28}{13}} - R^{\frac{28}{13}})$	Calcul. Press. Diff.	Actual Press. Diff.	Coeffic.	
							Calcul.	Actual
R	$\log R$	$\frac{28}{13}\log R$	$R^{\frac{28}{13}}$		$73.5(\ )$	p_d	$\frac{.298}{\sqrt{(\ )}}$	K
.1667	-.778	-1.68	.021	.165	12.15	12.15	.734	.730
.2060	-.686	-1.48	.033	.153	11.25	11.25	.762	.760
.2575	-.589	-1.27	.054	.132	9.75	9.75	.820	.820
.3150	-.502	-1.08	.083	.103	7.60	7.65	.928	.920
.3680	-.434	-.935	.116	.070	5.15	5.05	1.128	1.136
.458	-.339	-.730	.186					

Basis: $F = CR^{\overline{n}}$

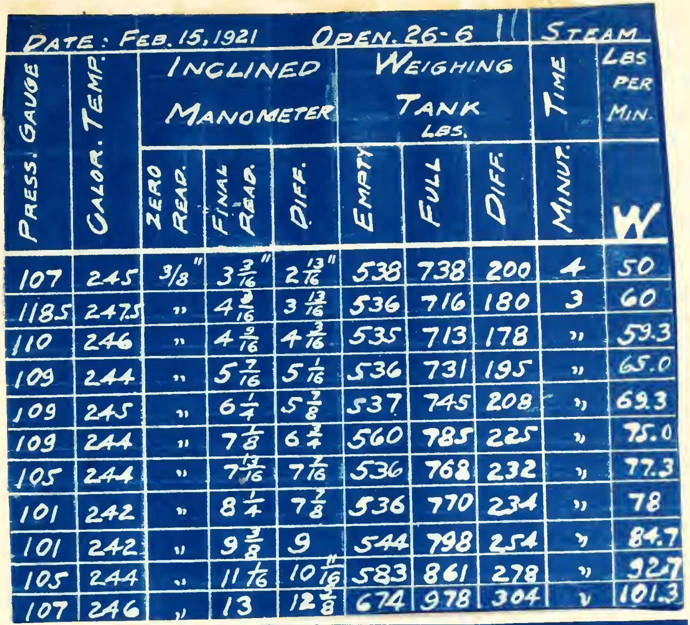
COMPARISON BETWEEN
CALCULATED AND ACTUAL
PRESSURES AND CO-EFFICIENTS

BASIS: $F = CR^{1/n}$

where D is the density corresponding to the absolute steam pressure as given by the steam tables, D is the actual density when corrected for quality, and X is the quality of the steam. The inclined mercury manometer with an inclination of 10° was used practically throughout the experiments on steam except for velocities below 4000 ft/min when the vertical manometer was used with acetylene tetra-chloride as the manometer fluid.

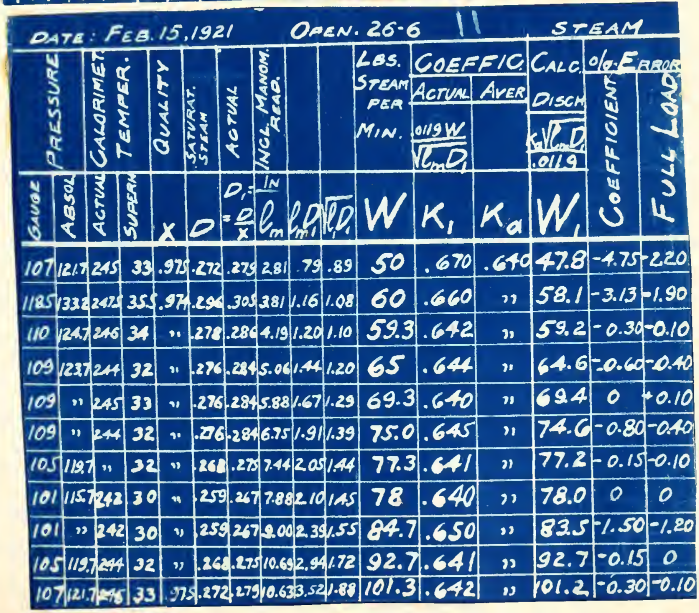
84. The values of the coefficients of connection 26-6 as calculated from all the data taken for *steam* flowing through the 3" hyperbolic test elbow, are plotted, in Fig. 14, as ordinates, with discharge in lbs./min, as abscissae. This figure also includes the "full load % error" and the "co-efficient % error" curves for the data of Tables 9 and 10. Just as in the case of water, there is a close agreement between the different values of the co-efficient with only small differences between any particular values and the average. The errors for the co-efficients of Table 5, have a maximum of 4¾% at one point, as indicated by the co-efficient % error curve, marked (2), the majority of the co-efficients having errors of less than 3%. If the errors are

TABLES 9 and 10



DATE: FEB. 15, 1921   OPEN. 26-6 || STEAM

PRESS. GAUGE	CALOR. TEMP.	INCLINED MANOMETER			WEIGHING TANK LBS.			TIME	LBS PER MIN.
		ZERO READ.	FINAL READ.	DIFF.	EMPTY	FULL	DIFF.	MINUT.	W
107	245	3/8"	3 3/16"	2 13/16"	538	738	200	4	50
118.5	247.5	"	4 3/16	3 13/16	536	716	180	3	60
110	246	"	4 9/16	4 3/16	535	713	178	"	59.3
109	244	"	5 7/16	5 1/16	536	731	195	"	65.0
109	245	"	6 1/4	5 7/8	537	745	208	"	69.3
109	244	"	7 1/8	6 3/4	560	785	225	"	75.0
105	244	"	7 13/16	7 7/16	536	768	232	"	77.3
101	242	"	8 1/4	7 7/8	536	770	234	"	78
101	242	"	9 3/8	9	544	798	254	"	84.7
105	244	"	11 1/16	10 11/16	583	861	278	"	92.7
107	246	"	13	12 5/8	674	978	304	"	101.3



DATE: FEB. 15, 1921   OPEN. 26-6 || STEAM

PRESSURE		CALORIMET. TEMPER.		QUALITY	SATURAT. STEAM	ACTUAL	INCL. MANOM. READ.			LBS. STEAM PER MIN.	COEFFIC. ACTUAL $\frac{.0119W}{\sqrt{l_m D_1}}$	COEFFIC. AVER	CALC. DISCH $\frac{K_a\sqrt{l_m D_1}}{.0119}$	% ERROR COEFFICIENT	% ERROR FULL LOAD
GAUGE	ABSOL	ACTUAL	SUPERH	x	D	$D_1 = \frac{D}{x}$	l_m IN	$l_m D_1$	$\sqrt{l_m D_1}$	W	K_1	K_a	W_1		
107	121.7	245	33	.975	.272	.279	2.81	.79	.89	50	.670	.640	47.8	-4.75	-2.20
118.5	133.2	247.5	35.5	.974	.296	.305	3.81	1.16	1.08	60	.660	"	58.1	-3.13	-1.90
110	124.7	246	34	"	.278	.286	4.19	1.20	1.10	59.3	.642	"	59.2	-0.30	-0.10
109	123.7	244	32	"	.276	.284	5.06	1.44	1.20	65	.644	"	64.6	-0.60	-0.40
109	"	245	33	"	.276	.284	5.88	1.67	1.29	69.3	.640	"	69.4	0	+0.10
109	"	244	32	"	.276	.284	6.75	1.91	1.39	75.0	.645	"	74.6	-0.80	-0.40
105	119.7	"	32	"	.268	.275	7.44	2.05	1.44	77.3	.641	"	77.2	-0.15	-0.10
101	115.7	242	30	"	.259	.267	7.88	2.10	1.45	78	.640	"	78.0	0	0
101	"	242	30	"	.259	.267	9.00	2.39	1.55	84.7	.650	"	83.5	-1.50	-1.20
105	119.7	244	32	"	.268	.275	10.69	2.94	1.72	92.7	.641	"	92.7	-0.15	0
107	121.7	245	33	.975	.272	.279	10.63	3.52	1.88	101.3	.642	"	101.2	-0.30	-0.10

figured on a basis of the difference between calculated flow and actual flow, compared with full load flow, these errors are still further reduced as illustrated by curve (3) marked "full load % error" curve. The maximum error in this case is only 2%, the majority of the errors being less than 1%.

85. In general, an inspection of all the data taken for steam indicates that the maximum error, on a full load percentage basis, is about 3½%, about 75% of all the errors being less than 2%. On a basis of co-efficient % error or of instantaneous rather than full load, the maximum error is about 5%, the great majority of errors being less than 3%.

PRESSURE DISTRIBUTION FOR STEAM

86. In making out the table for pressure distribution along the line of symmetry, when steam flows through the 3" hyperbolic elbow, the pressures were all taken for a discharge rate of 75 lbs/min. To reduce all the nearest available data to a 75 lbs.min basis and to the same density, which was chosen as .275 lbs./cu.ft., the relation

$$p = C \frac{W^2}{D} \qquad \text{(see eq. 34)}$$

was used. Thus, the nearest data for 26-6 happened to be a discharge of, 84 #/min and a density of

.284 lbs/cu.ft., the inclined manometer reading being 6.75". The corrected manometer reading is

$$l_m = 6.75\frac{(75)^2}{74^2} \times \frac{.284}{.275} = \underline{6.975''}$$

and the corresponding pressure is

$$p = 11.4\, l_m = 11.4 \times 6.975 = 79.5\ \#/\text{sq.ft.}$$

Similar corrections were made for the other openings along the line of symmetry.

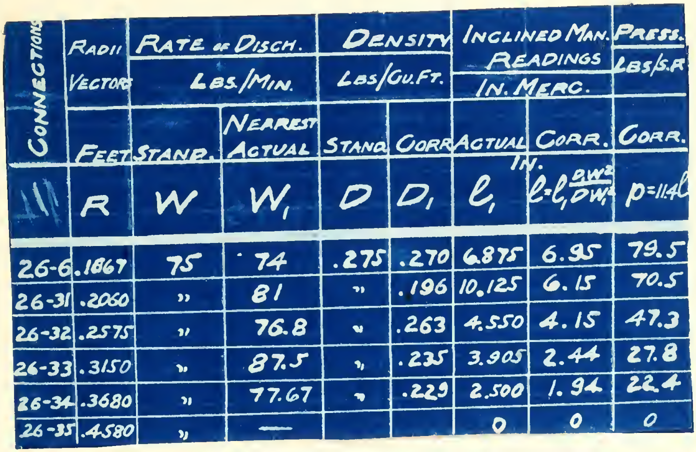
87. Table 11 gives the pressure distribution along the line of symmetry for steam, all data being reduced to a basis of 75 lbs/min, discharge, and a density of .275 lbs/ cu.ft., as just explained.

88. The pressure distribution curve along the line of symmetry of a 3" hyperbolic elbow when steam flows through it, is shown in Fig. 15 below. This curve was plotted from the data given in Table 11.

89. An inspection of the pressure distribution curve seems to suggest that the curve is represented by some periodic function such as a sine function. The x-axis of this sine curve appears to be parallel to the straight line CF connecting the two extreme points of the pressure curve. If two straight lines are drawn parallel to CF and tangent to the pressure curve, then the

TABLE 11

STEAM



Connections	Radii Vectors	Rate of Disch. Lbs./Min.		Density Lbs/Cu.Ft.		Inclined Man. Readings In. Merc.		Press. Lbs/S.F
	Feet	Stand.	Nearest Actual	Stand.	Corr.	Actual	Corr. In.	Corr.
	R	W	W_1	D	D_1	l_1	$l=l_1\frac{DW^2}{DW_1^2}$	$p=11.4l$
26-6	.1867	75	74	.275	.270	6.875	6.95	79.5
26-31	.2060	"	81	"	.196	10.125	6.15	70.5
26-32	.2575	"	76.8	"	.263	4.550	4.15	47.3
26-33	.3150	"	87.5	"	.235	3.905	2.44	27.8
26-34	.3680	"	77.67	"	.229	2.500	1.94	22.4
26-35	.4580	"	—			0	0	0

PRESSURE DISTRIBUTION
CURVE DATA
FOR
LINE OF SYMMETRY.

x-axis of the sine curve must lie midway between and parallel to these two lines. The x-axis is, therefore, the line ADEB. We shall take as the y-axis a line perpendicular to the x-axis at point A, i.e., at the intersection of the x-axis with the p-axis.

90. Therefore, if referred to the x-y axis just mentioned, the general equation of the pressure curve may be written as,

$$y = n\sin\ (mx+c) \qquad (56)$$

where n, m and c are the constants determining the relative position and exact nature of the sine curve.

91. If now, the x-y co-ordinates are turned through the angle, θ, so as to make the new x-axis parallel to the R-axis (see Fig.15) and so as to make the y-axis coincide with the p-axis, the following substitutions must be made for y and x in equation (56)

$$y = x'\sin\theta + y\ \cos\theta \qquad (57)$$

$$x = x'\ \cos\theta - y'\ \sin\theta \qquad (58)$$

If in addition to that, the x-axis is lowered a distance, y_0, until it coincides with the R_1-axis, we must write,

$$y' = y'' - y_0 \qquad (59)$$

or equations (57) and (58) must be changed to

$$y = x' \sin \theta + (y''-y_0) \cos \theta \qquad (57a)$$

$$x = x' \cos \theta - (y'' -y_0) \qquad (58a)$$

As a result, the new equation of the pressure curve when referred to co-ordinates coinciding with the p-R axes, becomes:

$$x' \sin \theta - (y''-y_0) \cos \theta = n \sin \left\{ m\left[x' \cos \theta - (y''-y_0) \sin \theta\right] + c \right\} \qquad (60)$$

92. But from the scales used for the p and R axis and from the general details of the curve, we have,

$$x' = .25R \quad (x' = \text{inches}) \qquad (61a)$$

$$y'' = .1\,p \quad (y'' = \text{''}\ ) \qquad (61b)$$

$$y_o = .1\,p_o = .1 \times 120 = 12 \qquad (61c)$$

$$\tan \theta = \frac{y_o}{x_o} = \frac{120 \times .1}{44 \times .25} = 1.09; \quad \theta = 47^\circ\,28' \qquad (61d)$$

$$\cos \theta = \cos 47^\circ\,28' = .676 \qquad (61e)$$

$$\sin \theta = \sin 47^\circ\,28' = .736 \qquad (61f)$$

93. FURTHERMORE, from the sine curve, when referred to the original set of co-ordinates, we get at points O and O_1,

$y = o$ for $x_1 = 11/32''$ and

$y = o$ " $x_2 = 5\frac{3}{32}''$

or

$$y = n\sin (mx_1 + c) = n\sin(mx_2 + c) = 0 \qquad (62)$$

i.e.

$$mx_1 + c = \pi \quad \text{and} \qquad (63a)$$

$$mx_2 + c = 2\pi \qquad (63b)$$

from which by subtraction we get,

$$m\,(x_1 - x_2) = 2\pi - \pi = \pi \qquad (63c)$$

and,

$$m = \frac{\pi}{x_2 - x_1} = \frac{\pi}{5\ 3/32 - 1\,\frac{1}{32}} = \underline{.246\pi} = .\underline{775} \qquad (64)$$

Substituting this value of m into equation (63a) we get

$$.775\,x_1 + c = \pi \text{ or} \qquad (65)$$

$$C = \pi - .775\,x_1 = \pi - .775 \times 1\ 1/32 = \underline{.74\pi} =$$
$$= \underline{2.34} \qquad (65)$$

Finally, from the maximum co-ordinates of the sine - curve, we get,

$$y_{max} = \tfrac{1}{2} = n \sin 90 = n, \text{ or}$$
$$n = 1/2 = .\underline{5} \qquad (66)$$

94 We have then in the last two paragraphs the values of all the constants involved in equation (60). Substituting these values into equation (60) we get,

$$.736\,(.25R) + .676\,(.1p - 12) =$$
$$= .5 \sin \left\{ .775 \left[.676(.25R) - .736(.1p - 12) \right] + 2.34 \right\}$$

which when simplified becomes

$$18.4\,R + .0676\,(p - 120) = .5 \sin (13.1\,R - .057p + 9.17) \qquad (67)$$

95. To change the pressure equation as

given in (67) to the form $p = \phi(R)$, we may as an approximation, eliminate p from the sine expression by assuming for an instant that the average expression for p is represented by the inclined line ADEB i.e., the line of reference or x-axis of the sine-curve, CDEF. The equation of this line can be written at once if we note that it intersects the p-axis at $p = 120$, and that its inclination is

$$\frac{120}{.44} = 273$$

Thus, we have that approximately,

$$p = 120 - 273\,R \qquad (68)$$

We must bear in mind that this equation is only true for the line ADEB for the scales of p and R used in plotting the pressure curve. It is not the true equation if uniform scales were used on the p and R axes. However, since it is the relation between p and R that is sought, equation (68) is evidently the proper one for the purpose.

96. By substituting the approximate value of p as given by equation (68) into the sine expression of equation (67) we get,

$$\sin(13.1\,R - .057\,p + 9.17) =$$
$$= \sin\left[13.1\,R - .057\,(120 - 273R) + 9.17\right] =$$
$$\sin(28.5\,R + 2.32) \qquad (69)$$

Therefore, we may simplify equation (67) as follows:

$$18.4\ R + .0676\ (p-120) = .5 \sin\ (28.5\ R + 2.32)$$

or

$$p = 120 - \frac{18.4\ R}{.0676} + \frac{.5}{.0676} \sin\ (28.5R+2.32) =$$

$$= 120 - 273\ R + 7.4. \sin\ (28.5\ R + 2.32) \qquad (70)$$

97. To check the accuracy of the derivation of the last equation as well as to find the errors caused by the approximations assumed in changing the pressure equation to the form,

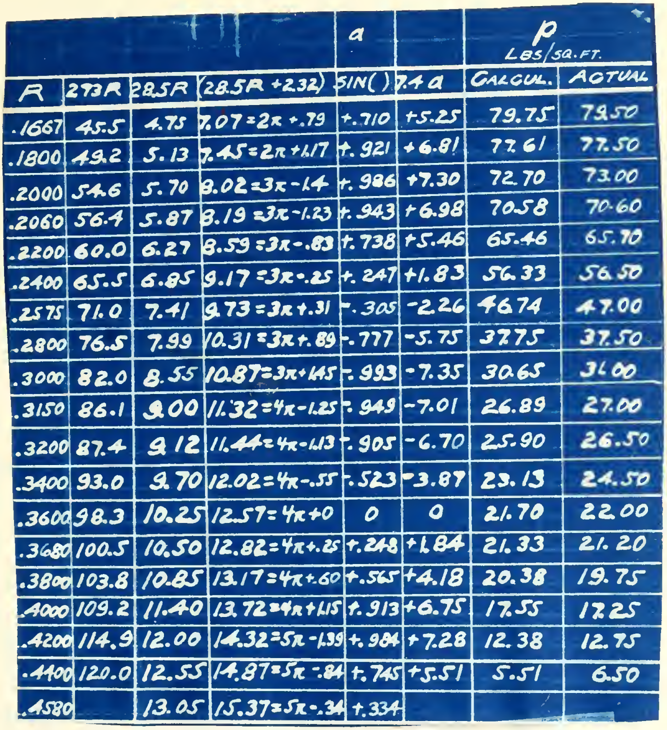
$$p = \phi\ (R)$$

calculated values of p, based upon equation (70) are compared with actual values of **p**, in Table 12, below. Evidently the agreement between the calculated and the actual values of pressure found in Table 12 is satisfactory enough for most practical purposes. If a still better agreement were desired, we could substitute the value of p as given in equation (70) into the original sine expression, involving p, of equation (67), Thus,

$$\begin{aligned}&\sin\ (13.1\ R - .057p + 9.17) = \\ &\sin \left\{ 13.1\ R - .057\left[120-273\ R + 7.4 \sin\ (28.5\ R + .2.32)\right] + 9.17 \right\} =\end{aligned}$$

TABLE 12

STEAM



				a		p LBS/SQ.FT.	
R	273R	28.5R	(28.5R +2.32)	SIN( )	7.4 a	CALCUL.	ACTUAL
.1667	45.5	4.75	7.07 = 2π + .79	+.710	+5.25	79.75	79.50
.1800	49.2	5.13	7.45 = 2π + 1.17	+.921	+6.81	77.61	77.50
.2000	54.6	5.70	8.02 = 3π − 1.4	+.986	+7.30	72.70	73.00
.2060	56.4	5.87	8.19 = 3π − 1.23	+.943	+6.98	70.58	70.60
.2200	60.0	6.27	8.59 = 3π − .83	+.738	+5.46	65.46	65.70
.2400	65.5	6.85	9.17 = 3π − .25	+.247	+1.83	56.33	56.50
.2575	71.0	7.41	9.73 = 3π + .31	−.305	−2.26	46.74	47.00
.2800	76.5	7.99	10.31 = 3π + .89	−.777	−5.75	37.75	37.50
.3000	82.0	8.55	10.87 = 3π + 1.45	−.993	−7.35	30.65	31.00
.3150	86.1	9.00	11.32 = 4π − 1.25	−.949	−7.01	26.89	27.00
.3200	87.4	9.12	11.44 = 4π − 1.13	−.905	−6.70	25.90	26.50
.3400	93.0	9.70	12.02 = 4π − .55	−.523	−3.87	23.13	24.50
.3600	98.3	10.25	12.57 = 4π + 0	0	0	21.70	22.00
.3680	100.5	10.50	12.82 = 4π + .25	+.248	+1.84	21.33	21.20
.3800	103.8	10.85	13.17 = 4π + .60	+.565	+4.18	20.38	19.75
.4000	109.2	11.40	13.72 = 4π + 1.15	+.913	+6.75	17.55	17.25
.4200	114.9	12.00	14.32 = 5π − 1.39	+.984	+7.28	12.38	12.75
.4400	120.0	12.55	14.87 = 5π − .84	+.745	+5.51	5.51	6.50
.4580		13.05	15.37 = 5π − .34	+.334			

COMPARISON
between the

PRESSURE CURVE AND CALCULATED PRESSURES.

$= \sin\left[28.5\,R + 2.32 - .421 \sin(28.5R + 2.32)\right]$

Then, the expression for p would become,

$$p = 120 - 273R + 7.4 \sin\left[28.5R + 2.32 - .421 \sin(28.5R + 2.32)\right] \qquad (71)$$

But this was considered hardly necessary, considering the results of Table 12.

"F" CURVE FOR STEAM

98. In the case of steam, because of the familiar appearance of the pressure distribution curve (Fig.15) as compared with the more or less unfamiliar nature of the "F" curve (Fig.16) it was found more convenient to first find the equation for the pressure curve and then to find the expression for the "F" curve from the equation,

$$\frac{dp}{dR} = -\frac{D}{g} RF^2 \qquad \text{(see Eq. 20)}$$

This method is evidently the reverse of that used in the case of water. Then, it was easier to determine by inspection the general nature of the "F" curve rather than that of the pressure curve since that particular "F" curve could be seen to be either an approximate straight line or a higher parabolic curve.

99. For steam, therefore, we get differentiating the pressure equation (70) with respect to R

$$\frac{dp}{dR} = -273 + 211 \cos(28.5\,R + 2.32) \qquad (72)$$

Using the latter expression for $\frac{dp}{dR}$, the equation of the "F" curve can be found from the equation

$$F = \sqrt{\frac{-g}{D}\,\frac{1}{R}\,\frac{dp}{dR}} \qquad \text{(Eq.20 solved for } \underline{F}\text{)}$$

viz.,

$$F = \sqrt{\frac{-32.2}{.275\ R}\left[-273 - 211\cos(28.5R + 2.32)\right]}$$

$$= \frac{179}{\sqrt{R}}\sqrt{1 - .775\cos(28.5\ R + 2.32)} =$$

$$= \frac{B\sqrt{1-.775\cos(28.5R + 2.32)}}{\sqrt{R}} \qquad (73)$$

100. Table 13 gives the calculations and values of the ordinates of the "F" curve for steam as obtained from the last equation, viz.eq. (73).

101. The equation for velocity of steam may now be written by substituting the expression for "F", just found, into the equation,

$$v = R\ F \qquad \text{(see eq. 19)}$$

Thus, we get,

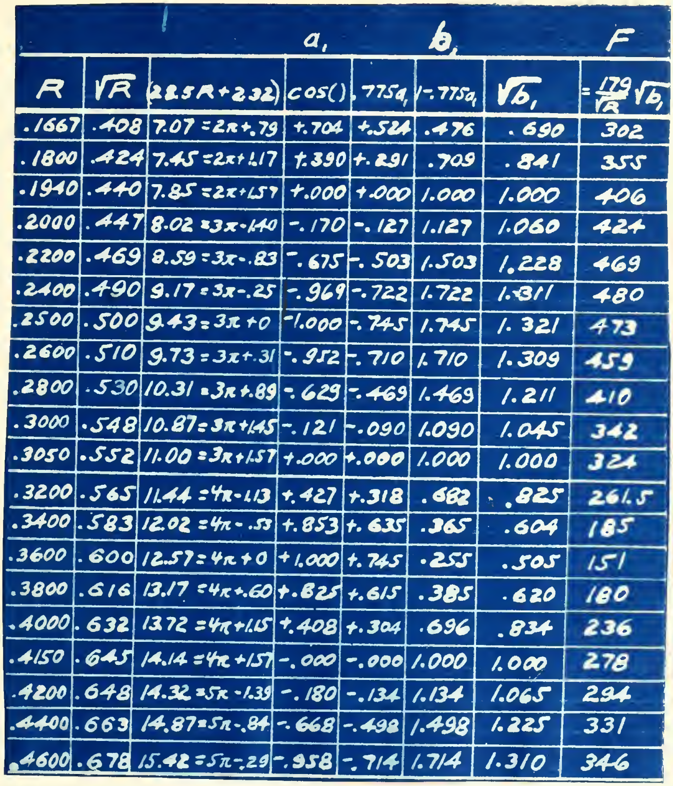
$$v = RF = 179\sqrt{R}\ \sqrt{1-.775\cos(28.5R + 2.32)} =$$

$$= B\sqrt{R}\ \sqrt{1-.775\cos(28.5 + 2.32)} \qquad (74)$$

102. To compare the velocity distri-

TABLE 13

STEAM



			a_1		b_1		F
R	$\sqrt{R}$	(285R+232)	cos()	$.775a_1$	$1-.775a_1$	$\sqrt{b_1}$	$=\frac{179}{\sqrt{R}}\sqrt{b_1}$
.1667	.408	7.07 = 2π+.79	+.704	+.524	.476	.690	302
.1800	.424	7.45 = 2π+1.17	+.390	+.291	.709	.841	355
.1940	.440	7.85 = 2π+1.57	+.000	+.000	1.000	1.000	406
.2000	.447	8.02 = 3π-1.40	-.170	-.127	1.127	1.060	424
.2200	.469	8.59 = 3π-.83	-.675	-.503	1.503	1.228	469
.2400	.490	9.17 = 3π-.25	-.969	-.722	1.722	1.311	480
.2500	.500	9.43 = 3π+0	-1.000	-.745	1.745	1.321	473
.2600	.510	9.73 = 3π+.31	-.952	-.710	1.710	1.309	459
.2800	.530	10.31 = 3π+.89	-.629	-.469	1.469	1.211	410
.3000	.548	10.87 = 3π+1.45	-.121	-.090	1.090	1.045	342
.3050	.552	11.00 = 3π+1.57	+.000	+.000	1.000	1.000	324
.3200	.565	11.44 = 4π-1.13	+.427	+.318	.682	.825	261.5
.3400	.583	12.02 = 4π-.53	+.853	+.635	.365	.604	185
.3600	.600	12.57 = 4π+0	+1.000	+.745	.255	.505	151
.3800	.616	13.17 = 4π+.60	+.825	+.615	.385	.620	180
.4000	.632	13.72 = 4π+1.15	+.408	+.304	.696	.834	236
.4150	.645	14.14 = 4π+1.57	-.000	-.000	1.000	1.000	278
.4200	.648	14.32 = 5π-1.39	-.180	-.134	1.134	1.065	294
.4400	.663	14.87 = 5π-.84	-.668	-.498	1.498	1.225	331
.4600	.678	15.42 = 5π-.29	-.958	-.714	1.714	1.310	346

"F" CURVE

DATA and CALCULATIONS

bution for steam with that for water, the values for both were calculated for different values of R and tabulated in Table 14 below.

Equation (46) viz.,

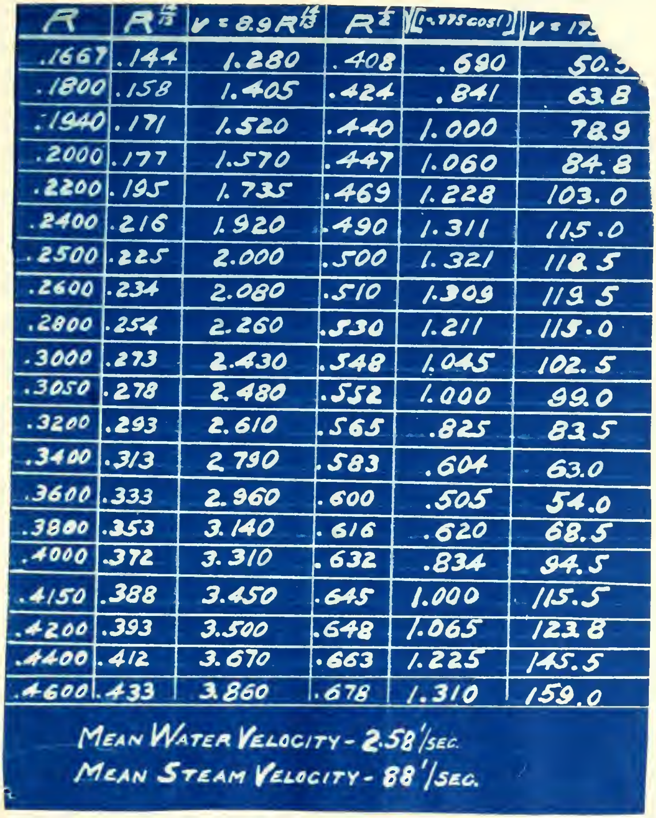
$$v = C\,R^{14/13}$$

was used for calculating the velocity distribution for water, using the value $C = 8.9$, found in equation (45), this value of C corresponding to the particular velocity of water, viz., 2.58 ft/sec., used for the pressure distribution analysis.

103. Figure 16 shows the "F" curve for steam, as plotted from the data of Table 13.

104. Fig. 17 shows a comparison between the velocity distribution curves for steam and water, indicating that the effect of friction is much more pronounced in the case of steam than in the case of water. The reason for this is probably the fact that internal friction depends not only upon the co-efficient of viscosity μ but also upon the density, ρ, or to be more exact, upon the ratio $\frac{\mu}{\rho}$ The value of μ for water in C.G.S. units is about $\mu_{water} = .01$, at ordinary room temperatures.

TABLE 14
STEAM - WATER



R	$R^{14/13}$	$v = 8.9R^{14/13}$	$R^{1/2}$	$\sqrt{[1-.775\cos(\,)]}$	$v = 17$
.1667	.144	1.280	.408	.690	50.
.1800	.158	1.405	.424	.841	63.8
.1940	.171	1.520	.440	1.000	78.9
.2000	.177	1.570	.447	1.060	84.8
.2200	.195	1.735	.469	1.228	103.0
.2400	.216	1.920	.490	1.311	115.0
.2500	.225	2.000	.500	1.321	118.5
.2600	.234	2.080	.510	1.309	119.5
.2800	.254	2.260	.530	1.211	115.0
.3000	.273	2.430	.548	1.045	102.5
.3050	.278	2.480	.552	1.000	99.0
.3200	.293	2.610	.565	.825	83.5
.3400	.313	2.790	.583	.604	63.0
.3600	.333	2.960	.600	.505	54.0
.3800	.353	3.140	.616	.620	68.5
.4000	.372	3.310	.632	.834	94.5
.4150	.388	3.450	.645	1.000	115.5
.4200	.393	3.500	.648	1.065	123.8
.4400	.412	3.670	.663	1.225	145.5
.4600	.433	3.860	.678	1.310	159.0

MEAN WATER VELOCITY - 2.58'/SEC.
MEAN STEAM VELOCITY - 88'/SEC.

COMPARATIVE VELOCITY DISTRIBUTION
FOR

WATER AND STEAM

The value of μ for steam, to the best knowledge of the writer, has never yet been found, but as in the case of all gases and vapors it is probably much smaller than that for liquids. Thus the value of μ for air is, according to Maxwell,

$$\mu = .000187 \left(1 + \frac{t}{273}\right) =$$

$$= .6875 \times 10^{-6} T \qquad (75)$$

(See Lamb's Hydrodynamics)

p. 219

where t = °C, and T is the absolute temperature using the centrigrate scale. Then for steam temperatures, say, 350° F = 5/9 (350-32) °C = 176° C, we get,

$$\mu \text{ air} = .6875 \times 10^{-6} (176 + 273) =$$

$$= 3.43 \times 10^{-4}$$

which is,

$$\frac{\mu \text{ water}}{\mu \text{ air}} = \frac{.01}{3.43 \times 10^{-4}} = 29.2$$

times smaller than that for water. If the viscosity of steam were of the same order ~~for~~ of magnitude as that we just calculated for air, the fact, that the density of the steam is so much smaller than that of water would more than offset the effect of smaller viscosity for steam. Thus, the ratios of ___ for the two compare as follows:

$$\frac{\frac{\mu}{\rho}\text{ steam}}{\frac{\mu}{\rho}\text{ water}} = \frac{62.5}{.25} \qquad \frac{1}{29.2} =$$

$$= 8.57$$

i.e., the effect of internal friction for steam would be between 8 and 9 times greater than that for water, if the co-efficient of viscosity for steam were of the same order of magnitude as that for air.

105. To calculate the expression for average velocity from equation (74), it is necessary to integrate the term

$$\sqrt{R} \qquad \sqrt{1- .775 \cos (28.5 R + 2.32)}$$

To the best knowledge of the writer, this term can not be integrated by any regular methods. For all practical purposes, however, mechanical integration by the use of planimeters, is accurate enough; for this reason, the author used a rectangular co-ordinate planimeter with an integral curve drawing attachment, going over a specially plotted velicity curve from R=0 to R = 1 foot.

106. Fig. 18 is the integral of the velocity curve as drawn by the rectangular co-ordinate planimeter. The abscissae are marked in feet as well as inches, while the ordinates are equal to

$$\theta (R) = \int \sqrt{R} \qquad \sqrt{1-.775 \cos (28.5 R + 2.32)}\, dR$$

$$(R) = \text{Feet}) \qquad (76)$$

We shall refer to this integral, hereafter, as the "θ" function.

107. Table 15 gives the calculations of the numerical values of

$$\sqrt{R} \qquad \sqrt{1- .775 \cos (28.5 R + 2.32)}$$

as well as the values of the "θ" function, viz.,

$$\theta (R) = \int \sqrt{R} \quad \sqrt{1-.775 \cos (28.5 + 2.32)} \; d R,$$

for the range shown. The former were used for plotting the velocity curve for values of R from 0 to 1 foot. The latter were obtained from the "θ" curve shown in Fig. 18, necessary corrections having been made for the coordinate scales used.

108. We now can find the expression for average velocity. Thus, from equation, (74) we have,

$$v = B\sqrt{R} \quad \sqrt{1-.775 \cos (28.5 R + 2.32)}$$

and

$$v_a = \frac{\int_{R_1}^{R_2} v \, dR}{R_2 - R_1} = \frac{B \left[\theta (R_2) - \theta (R_1)\right]}{R_2 - R_1} \qquad (77)$$

since

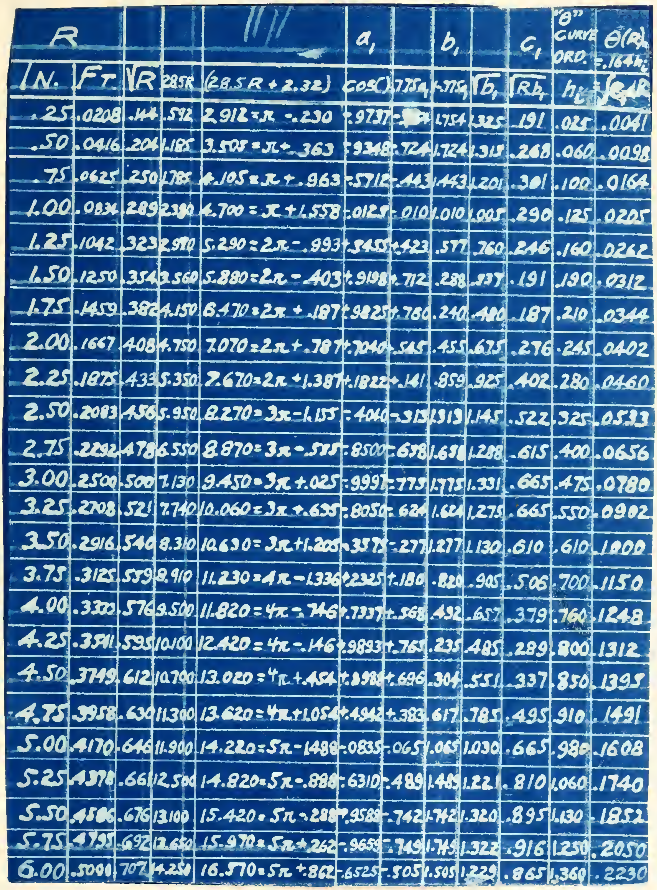
$$\int_{R_1}^{R_2} v \, dR = \int_{R_1}^{R_2} B\sqrt{R} \quad \sqrt{1 - .775 \cos (28.5R + 2.32)} \quad dR$$

$$= B \left[\theta (R_2) - \theta (R_1) \right]$$

according to equation (76).

TABLE 15

STEAM



R					a_1		b_1		c_1	"θ" CURVE ORD.	$\theta(R) = .164h_i$
IN.	FT.	$\sqrt{R}$	28.5R	$(28.5R + 2.32)$	cos()	$.775a_1$	$1-.775a_1$	$\sqrt{b_1}$	$\sqrt{Rb_1}$	h_i	$\int c_1 dR$
.25	.0208	.144	.592	$2.912 = \pi - .230$	−.9737	[illegible]	1.754	1.325	.191	.025	.0041
.50	.0416	.204	1.185	$3.505 = \pi + .363$	−.9348	−.724	1.724	1.315	.268	.060	.0098
.75	.0625	.250	1.785	$4.105 = \pi + .963$	−.5712	−.443	1.443	1.201	.301	.100	.0164
1.00	.0834	.289	2.390	$4.700 = \pi + 1.558$	−.0125	−.010	1.010	1.005	.290	.125	.0205
1.25	.1042	.323	2.970	$5.290 = 2\pi - .993$	+.5455	+.423	.577	.760	.246	.160	.0262
1.50	.1250	.354	3.560	$5.880 = 2\pi - .403$	+.9198	+.712	.288	.537	.191	.190	.0312
1.75	.1459	.382	4.150	$6.470 = 2\pi + .187$	+.9825	+.760	.240	.490	.187	.210	.0344
2.00	.1667	.408	4.750	$7.070 = 2\pi + .787$	+.7040	+.545	.455	.675	.276	.245	.0402
2.25	.1875	.433	5.350	$7.670 = 2\pi + 1.387$	+.1822	+.141	.859	.925	.402	.280	.0460
2.50	.2083	.456	5.950	$8.270 = 3\pi - 1.155$	−.4040	−.313	1.313	1.145	.522	.325	.0533
2.75	.2292	.478	6.550	$8.870 = 3\pi - .575$	−.8500	−.658	1.658	1.288	.615	.400	.0656
3.00	.2500	.500	7.130	$9.450 = 3\pi + .025$	−.9997	−.775	1.775	1.331	.665	.475	.0780
3.25	.2708	.521	7.740	$10.060 = 3\pi + .635$	−.8050	−.624	1.624	1.275	.665	.550	.0902
3.50	.2916	.540	8.310	$10.630 = 3\pi + 1.205$	−.3575	−.277	1.277	1.130	.610	.610	.1000
3.75	.3125	.559	8.910	$11.230 = 4\pi - 1.336$	+.2325	+.180	.820	.905	.506	.700	.1150
4.00	.3330	.576	9.500	$11.820 = 4\pi - .746$	+.7337	+.568	.432	.657	.379	.760	.1248
4.25	.3541	.595	10.100	$12.420 = 4\pi - .146$	+.9893	+.765	.235	.485	.289	.800	.1312
4.50	.3749	.612	10.700	$13.020 = 4\pi + .454$	+.8988	+.696	.304	.551	.337	.850	.1395
4.75	.3958	.630	11.300	$13.620 = 4\pi + 1.054$	+.4942	+.383	.617	.785	.495	.910	.1491
5.00	.4170	.646	11.900	$14.220 = 5\pi - 1.488$	−.0835	−.065	1.065	1.030	.665	.980	.1608
5.25	.4378	.661	12.500	$14.820 = 5\pi - .888$	−.6310	−.489	1.489	1.221	.810	1.060	.1740
5.50	.4586	.676	13.100	$15.420 = 5\pi - .288$	−.9588	−.742	1.742	1.320	.895	1.130	.1852
5.75	.4795	.692	13.650	$15.970 = 5\pi + .262$	−.9659	−.749	1.749	1.322	.916	1.250	.2050
6.00	.5000	.707	14.250	$16.570 = 5\pi + .862$	−.6525	−.505	1.505	1.229	.865	1.360	.2230

TABLE 15

STEAM CONT'D

R					a_1		b_1		c_1	"θ" CURVE ORDIN.	$\theta(R) = .164h_i$
IN.	FT.	$\sqrt{R}$	28.5R	(28.5R + 2.32)	cos()	$.775a_1$	$1-.775a_1$	$\sqrt{b_1}$	$\sqrt{Rb_1}$	h_i	$= \int q\,dR$
6.25	.521	.722	14.85	17.17 = 5π+1.462	-.1095	-.085	1.085	1.041	.751	1.470	.2410
6.50	.542	.738	15.45	17.77 = 6π-1.08	+.4712	+.365	.635	.796	.588	1.550	.2542
6.75	.563	.751	16.05	18.37 = 6π-.480	+.887	+.687	.313	.560	.421	1.620	.2660
7.00	.584	.765	16.62	18.94 = 6π+.090	+.996	+.771	.229	.478	.366	1.690	.2770
7.25	.605	.779	17.25	19.57 = 6π+.720	+.752	+.582	.418	.646	.503	1.760	.2885
7.50	.625	.790	17.80	20.12 = 6π+1.27	+.297	+.230	.770	.876	.692	1.800	.2950
7.75	.646	.805	18.40	20.72 = 7π-1.27	-.297	-.230	1.230	1.110	.895	1.900	.3120
8.00	.667	.816	19.05	21.37 = 7π-.62	-.814	-.630	1.630	1.275	1.040	2.010	.3300
8.25	.688	.829	19.60	21.92 = 7π-.07	-.998	-.773	1.773	1.330	1.100	2.110	.3480
8.50	.709	.841	20.20	22.52 = 7π + .53	-.863	-.668	1.668	1.291	1.085	2.250	.3690
8.75	.730	.844	20.80	23.12 = 7π+1.13	-.427	-.331	1.331	1.152	.972	2.375	.3900
9.00	.750	.865	21.40	23.72 = 8π-1.413	+.157	+.122	.878	.935	.810	2.480	.4070
9.25	.771	.879	22.00	24.32 = 8π-.813	+.687	+.532	.468	.685	.602	2.580	.4235
9.50	.791	.890	22.50	24.82 = 8π-.313	+.952	+.736	.264	.514	.457	2.675	.4380
9.75	.813	.900	23.15	25.47 = 8π+.337	+.944	+.730	.270	.520	.469	2.720	.4[illegible]60
10.00	.834	.913	23.75	26.07 = 8π+.937	+.593	+.459	.541	.736	.672	2.800	.4600
10.25	.855	.925	24.38	26.70 = 8π+1.57	0	0	1.000	1.000	.925	2.900	.4760
10.50	.875	.935	24.95	27.27 = 9π-1.00	-.540	-.418	1.418	1.189	1.111	3.000	.4920
10.75	.896	.946	25.55	27.87 = 9π-.40	-.921	-.714	1.714	1.310	1.240	3.100	.5085
11.00	.917	.955	26.10	28.42 = 9π+.15	-.989	-.765	1.765	1.330	1.270	3.220	.5280
11.25	.938	.967	26.70	29.02 = 9π+.75	-.731	-.566	1.566	1.251	1.210	3.375	.5540
11.50	.959	.979	27.30	29.62 = 9π+1.35	-.219	-.170	1.170	1.082	1.060	3.500	.5740
11.75	.980	.990	27.95	30.27 = 10π-1.15	+.409	+.317	.683	.826	.818	3.650	.6040
12.00	1.000	1.000	28.50	30.82 = 10π-.60	+.825	+.640	.360	.600	.600	3.800	.6240

"θ" CURVE
DATA AND CALCULATIONS.

109. Referring back to the fundamental relation,

$$\frac{dp}{dR} = -\frac{D}{g} \quad RF^2$$

we may evidently substitute for "F" its expression as given by equation (73), and then by integrating the new expression for $\frac{dp}{dR}$ find the equation for the pressure, p. Thus we have $\frac{dp}{dR} = -\frac{D}{g} \; R \; F^2$

$$= -\frac{D}{g} \quad R\frac{B^2}{R}\left[1-.775 \cos (28.5R + 2.32)\right] =$$

$$= -\frac{D}{g} \; B^2\left[1-.775 \cos (28.5 R - 2.32)\right] \quad (78)$$

from which,

$$p = p_o - \frac{D}{g} \; B^2\int\left[1-.775 \cos (28.5R + 2.32)\right] dR =$$

$$= p_0 - \frac{D}{g} B^2 R + .0272 \frac{D}{g} B^2 \sin (28.5 R + 2.32) \qquad (79)$$

110. The difference of pressure between any point, R, along the line of symmetry and point 26, whose radius vector is expressed by R, can now be written down by using equation (79) thus,

$$p - p_2 = \frac{D}{g} \; B^2\left\{(R_2-R) - .0272\left[\sin (28.5R + 2.32) - \right.\right.$$

$$\left.\left. - \sin (28.5 R_2 + 2.32)\right]\right\} =$$

$$= \frac{D}{g} B^2 \left[f\,(R_2) - f\,(R) \right] \qquad (80)$$

where f (R) is an abbreviation for the corresponding expression in the brackets.

111. Comparing the equation (80) for pressure difference with the equation

$$v_a = B \; \frac{\left[\theta(R_2) - \theta(R_1) \right]}{R_2 - R_1} \qquad \text{(see Eq.77)}$$

for average velocity, we note that as in previous cases, the constant, B is proportional to the average velocity, v_a, and to the square root of the pressure difference, $p - p_2$ between any two, points along the line of symmetry. We may, therefore, write,

$$v_a = B \; \frac{\left[\theta(R_2) - \theta(R_1) \right]}{R_2 - R_1} = \frac{\sqrt{\frac{2g\,(p-p_2)}{D}} \qquad \left[\theta(R_2) - \theta(R_1) \right]}{\sqrt{2\left[f(R_2) - f(R) \right]} \qquad (R_2 - R_1)} =$$

$$= \frac{\theta\,(R_2) - \theta\,(R_1)}{(R_2 - R_1)\sqrt{2\left[f(R_2) - f(R) \right]}} \sqrt{2\,g\,H} =$$

[illegible]

$$= K\sqrt{2g\,H} \qquad (81)$$

where,

$$K = \frac{\theta(R_2) - \theta(R_1)}{(R_2-R_1)\sqrt{2[f(R_2)-f(R)]}} \qquad (82)$$

is the steam co-efficient of any hyperbolic elbow whose **pressure** connections are made at points R, R_2 and whose minimum and maximum radii vectors are R_1 and R_2 respectively,

112. Values of

$$f(R) = R - .0272 \sin(28.5R + 2.32) \qquad (83)$$

were calculated and tabulated in Table 16 for values of R ranging from R = 0 to R = 1 foot. Therefore, for any hyperbolic elbow within the table range values of θ (R) and f (R) can at once be found from tables 15 and 16, respectively, and the flow meter co-efficient, K, can at once be accurately calculared from equation (82).

where,

$$K = \frac{\theta(R_2) - \theta(R_1)}{(R_2-R_1)\sqrt{2\left[f(R_2)-f(R)\right]}} \qquad (82)$$

is the steam co-efficient of any hyperbolic elbow whose pressure connections are made at points R, R_2 and whose minimum and maximum radii vectors are R_1 and R_2 respectively,

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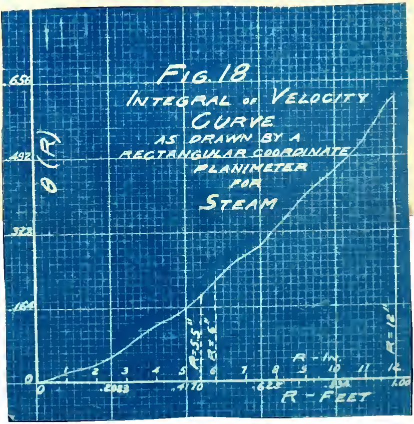
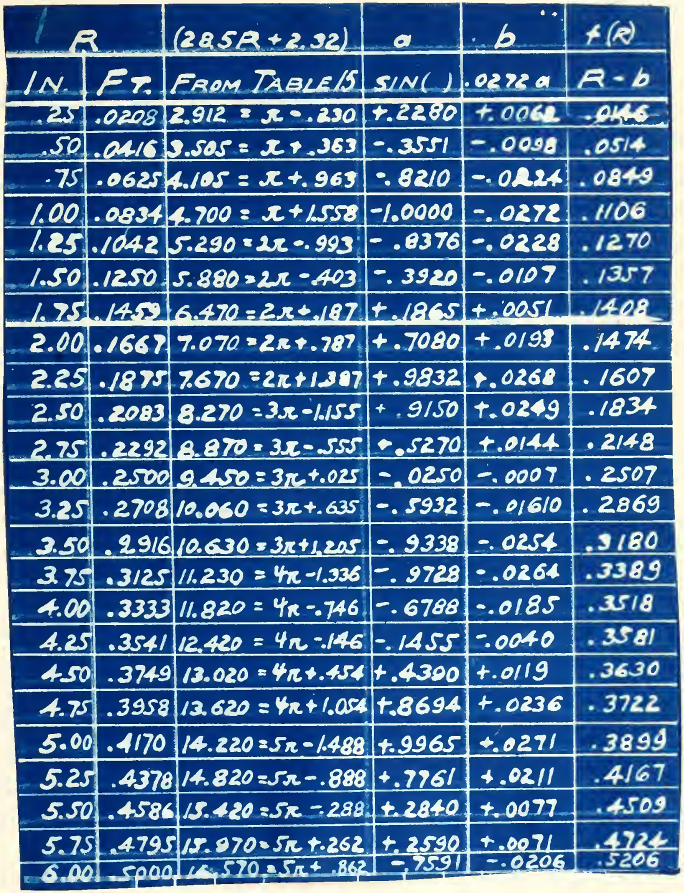


FIG. 18
INTEGRAL OF VELOCITY CURVE
AS DRAWN BY A RECTANGULAR COORDINATE PLANIMETER FOR STEAM

TABLE 16

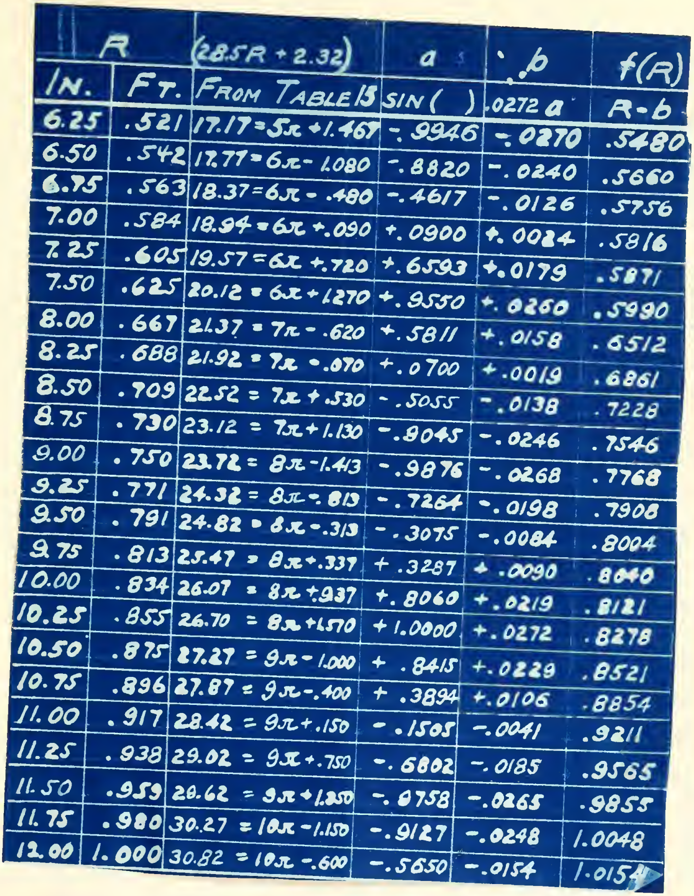
STEAM



R		(28.5R + 2.32)	a	b	f(R)
IN.	FT.	FROM TABLE 15	SIN( )	.0272 a	R - b
.25	.0208	2.912 = π - .230	+.2280	+.0062	.0146
.50	.0416	3.505 = π + .363	-.3551	-.0098	.0514
.75	.0625	4.105 = π + .963	-.8210	-.0224	.0849
1.00	.0834	4.700 = π + 1.558	-1.0000	-.0272	.1106
1.25	.1042	5.290 = 2π - .993	-.8376	-.0228	.1270
1.50	.1250	5.880 = 2π - .403	-.3920	-.0107	.1357
1.75	.1459	6.470 = 2π + .187	+.1865	+.0051	.1408
2.00	.1667	7.070 = 2π + .787	+.7080	+.0193	.1474
2.25	.1875	7.670 = 2π + 1.387	+.9832	+.0268	.1607
2.50	.2083	8.270 = 3π - 1.155	+.9150	+.0249	.1834
2.75	.2292	8.870 = 3π - .555	+.5270	+.0144	.2148
3.00	.2500	9.450 = 3π + .025	-.0250	-.0007	.2507
3.25	.2708	10.060 = 3π + .635	-.5932	-.01610	.2869
3.50	.2916	10.630 = 3π + 1.205	-.9338	-.0254	.3180
3.75	.3125	11.230 = 4π - 1.336	-.9728	-.0264	.3389
4.00	.3333	11.820 = 4π - .746	-.6788	-.0185	.3518
4.25	.3541	12.420 = 4π - .146	-.1455	-.0040	.3581
4.50	.3749	13.020 = 4π + .454	+.4390	+.0119	.3630
4.75	.3958	13.620 = 4π + 1.054	+.8694	+.0236	.3722
5.00	.4170	14.220 = 5π - 1.488	+.9965	+.0271	.3899
5.25	.4378	14.820 = 5π - .888	+.7761	+.0211	.4167
5.50	.4586	15.420 = 5π - .288	+.2840	+.0077	.4509
5.75	.4795	15.970 = 5π + .262	+.2590	+.0071	.4724
6.00	.5000	16.570 = 5π + .862	-.7591	-.0206	.5206

TABLE 16 CON'T

STEAM



R		(28.5R + 2.32)	a	b	f(R)
IN.	FT.	FROM TABLE 15	SIN ( )	.0272 a	R-b
6.25	.521	17.17 = 5π + 1.467	-.9946	-.0270	.5480
6.50	.542	17.77 = 6π - 1.080	-.8820	-.0240	.5660
6.75	.563	18.37 = 6π - .480	-.4617	-.0126	.5756
7.00	.584	18.94 = 6π + .090	+.0900	+.0024	.5816
7.25	.605	19.57 = 6π + .720	+.6593	+.0179	.5871
7.50	.625	20.12 = 6π + 1.270	+.9550	+.0260	.5990
8.00	.667	21.37 = 7π - .620	+.5811	+.0158	.6512
8.25	.688	21.92 = 7π - .070	+.0700	+.0019	.6861
8.50	.709	22.52 = 7π + .530	-.5055	-.0138	.7228
8.75	.730	23.12 = 7π + 1.130	-.9045	-.0246	.7546
9.00	.750	23.72 = 8π - 1.413	-.9876	-.0268	.7768
9.25	.771	24.32 = 8π - .813	-.7264	-.0198	.7908
9.50	.791	24.82 = 8π - .313	-.3075	-.0084	.8004
9.75	.813	25.47 = 8π + .337	+.3287	+.0090	.8040
10.00	.834	26.07 = 8π + .937	+.8060	+.0219	.8121
10.25	.855	26.70 = 8π + 1.570	+1.0000	+.0272	.8278
10.50	.875	27.27 = 9π - 1.000	+.8415	+.0229	.8521
10.75	.896	27.87 = 9π - .400	+.3894	+.0106	.8854
11.00	.917	28.42 = 9π + .150	-.1505	-.0041	.9211
11.25	.938	29.02 = 9π + .750	-.6802	-.0185	.9565
11.50	.959	29.62 = 9π + 1.350	-.9758	-.0265	.9855
11.75	.980	30.27 = 10π - 1.150	-.9127	-.0248	1.0048
12.00	1.000	30.82 = 10π - .600	-.5650	-.0154	1.0154

f (R)
DATE AND CALCULATIONS

113. For our 3" hyperbola, where

$$R_1 = 2'' = .1667' \text{ and } R_2 = 5.5'' = .4586'$$

Table 15 gives

$$\Theta\ (R_1) = .0402, \text{ and } \Theta\ (R_2) = .1852$$

or,

$$\Theta\ (R_2) - \Theta\ (R_1) = .1852 - .0402 = .1450$$

Also, Table 16 gives,

$$f\ (R_2) = .449$$

Therefore, the expression for the co-efficient becomes,

$$K = \frac{\Theta\ (R_2) - \Theta(R_1)}{(R_2 - R_1)\ \sqrt{2\left[f -(R_2) - f\ (R)\right]}}$$

$$= \frac{.1450}{(,4586 - .1667)\ \sqrt{2\left[.449 - f\ (R)\right]}} =$$

$$= \frac{.497}{\sqrt{.898 - 2f\ (R)}} \qquad (84)$$

114. The last equation is the general expression for the steam co-efficients of our 3" hyperbolic elbow, connections being made along the line of symmetry, and all pressures being compared with that at opening 26. This equation, as we trace its origin, is found to be the final result of various deductions from equation (70) for pressure distribution as re-

presented by the pressure curve of Fig. 15. We may now check the accuracy of the equations for the "F" curve, the velocity curve and the co-efficients by calculating the latter according to the equation (84), and then comparing the calculated with the actual co-efficients for connections,

26-6, 26-31, 26-32, 26-23, 26-34, respectively.

115. Table 17, below, shows how the calculated co-efficients compare with the actual co-efficients.

116. Another way of checking our results is to compare the mean ordinate of the velocity curve with the actual mean velocity. Thus, since the pressures were all taken at or reduced to a discharge of $W = 75$ lbs/min., and a density of $D = .275$, the actual mean velocity is,

$$v_a = .323 \frac{W}{D} = .323 \frac{75}{.275} = 88 \text{ ft/sec}$$

(See general Data)

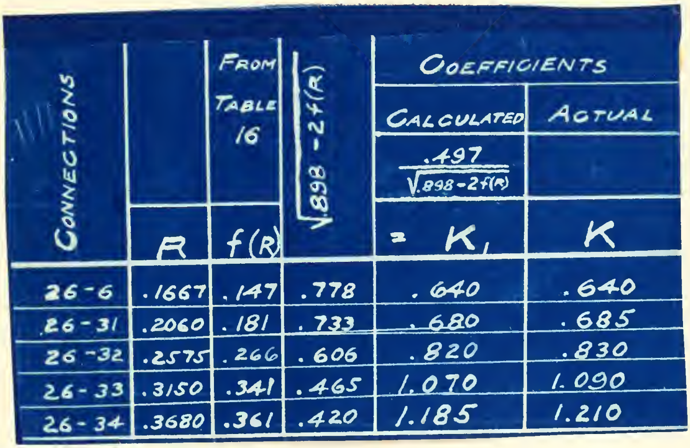
On the other hand the equation for the mean velocity is,

$$v_a = B \frac{\left[\theta(R_2) - \theta(R_1)\right]}{R_2 - R_1} \quad \text{(see Eq. 77)}$$

where $B = 179$ (see eq. 74)

TABLE 17

STEAM



Connections		From Table 16	$\sqrt{.898-2f(R)}$	Coefficients	
				Calculated	Actual
				$\frac{.497}{\sqrt{.898-2f(R)}}$	
	R	f(R)		$= K_1$	K
26-6	.1667	.147	.778	.640	.640
26-31	.2060	.181	.733	.680	.685
26-32	.2575	.266	.606	.820	.830
26-33	.3150	.341	.465	1.070	1.090
26-34	.3680	.361	.420	1.185	1.210

COMPARISON BETWEEN
CALCULATED AND ACTUAL COEFFICIENTS

$R_2 = .4586 \quad R_1 = .1667$, and

$\theta(R_2) - \theta(R_1) = .1450$ (see paragraph 113):

i.e., theoretically, we have,

$$v_a = 179 \times \frac{.145}{.2919} = 89 \text{ ft/sec.}$$

which differs but slightly from the actual mean velocity.

117. It is, therefore, apparent that there is general agreement between the calculated and the actual velocities and co-efficients, indicating the correctness of all the derived equations, at least for the range covered by the experimental investigation,

118. It may be remarked, that for estimating purposes, approximate values of the flow meter co-efficient, K, for steam may be calculated by assuming the velocity integral curve to be approximately a higher parabola of the form,

$$\theta(R) = a R^{\frac{3}{2}} \qquad (85)$$

and by assuming further that equation (80) may be approximated by

$$p - p_2 = \frac{D}{g} B^2 (R_2 - R) \qquad (86)$$

or

$f(R) = R$ (Compare with eq. 83)

Thus, from Table 15, we have,

$R = .813 \quad \theta(R) = .446 \quad$ or $a = \dfrac{\theta(R)}{R^{3/2}} =$

$$\frac{.446}{.813^{3/2}} = .61$$

$R = .250 \quad \theta(R) = .078 \quad$ or $a = \dfrac{.078}{.25^{3/2}} = .625$

$R = .584 \quad \theta(R) = .277$

or $a = \dfrac{\theta(R)}{R^{3/2}} = \dfrac{.277}{.584^{3/2}} = .620$

or we may assume as an approximate mean value,

$$C = .62$$

Then, the approximate expression for the flow meter co-efficient becomes,

$$K = \frac{\theta(R_2) - \theta(R_1)}{(R_2 - R_1)\sqrt{2\left[f(R_2) - f(R_1)\right]}} =$$

$$\frac{.62\left(R_2^{\frac{3}{2}} - R_1^{\frac{3}{2}}\right)}{(R_2 - R_1)\sqrt{2(R_2 - R_1)}} = \frac{.438\left(R_2^{3/2} - R_1^{3/2}\right)}{(R_2 - R_1)^{3/2}} \qquad (87)$$

Thus, for our 3" elbow,

$R_1 = .1667 \quad R_2 = .4586$, and

$$K = .438 \quad \frac{.4586^{3/2} - .1667^{3/2}}{.4586 - .1667} = \frac{.438 \times .242}{.1575} = \underline{.675}$$

which is within 95% accurate, or close enough as an approximate estimate.

CONCLUSIONS

We conclude from the results of this investigation that in the rectangular hyperbolic elbow with its straightening grids, we have at least one flow metering device of dependable commercial accuracy. Since the accuracy of this device in no way depends upon the nature of local piping complications, shop calibrations only are necessary. This is a feature which is very important from a practical point of view.

120. Because of the stream line flow in the hyperbolic elbow as against the complicated turbulent flow in the original elbow, before it is replaced by the hyperbolic type, there is a gratifying reduction in friction pressure drop. Thus, in ordinary piping and fittings the pressure drop is approximately proportional to the square of the velocity. For stream line flow, however, internal friction instead of "skin" friction predominates, and it is a fact that friction, produced by viscous forces alone are proportional to the first power of the velocity instead of the second power.

121. For example, in the case of capillary tubes, the relation between velocity and pressure drop is expressed by

$$v = p\frac{(R^2 - r^2)}{4\mu l} \qquad (88)$$

(See Lamb's Hydrodynamics)

p. 220-221

where,

R- = radius of the tube
v = velocity, at a distance, r, from the center,
$\frac{p}{l}$ = pressure drop per unit length of tube,

μ = co-efficient of viscosity of the fluid. The pressure drop per unit length p is thus seen to be proportional to the first power of the velocity; viz.,

$$\frac{p}{l} = \frac{4\mu v}{R^2 - r^2} = cv \qquad (89)$$

Or, if the mean velocity, v_a, corresponding to the total discharge Q, is considered, we have the relation,

$$Q = \frac{\pi R^4 p}{8\mu l} \qquad (90)$$

(Same reference as for eq. 88)

from which,

$$\frac{p}{l} = \frac{8\mu Q}{\pi R^4} = \frac{8\mu\pi R^2 V_a}{\pi R^4} = \frac{8\mu V_a}{R^2} = \qquad (91)$$

$$= C_1 V_a$$

or the pressure drop $\frac{p}{l}$, is proportional to the first power of the mean velocity $\underline{v_a}$.

122. In using the hyperbolic elbow as a flow metering device, just as in the case of the pitot tube, orifice, or venturi tube, it must be re-

membered that in the formula,

$$v = K\sqrt{2\ g\ H}$$

H represents head in feet of the particular fluid whose flow is to be measured and not pressure difference p. In fact, as was formerly shown,

$$H = \frac{p}{D}$$

where, p = pressure difference, lbs/ sq. ft.

and D = density of the fluid, lbs/cu.ft.

or we have,

$$v = K\sqrt{2g\frac{p}{D}} \qquad \text{(see eq. 16)}$$

Apparently, then, the density, D, of the fluid is of as much importance as the pressure difference, p, for calculating velocity of flow, v. Since, the density D in the case of steam, gases, and oils varies with pressure, temperature, nature, of oil, etc., it is important to have some means of correcting for these variations, if the flow meter is to be accurate.

123. When the flow is to be measured in weight per unit of time, say pounds per hour, we note that the expression for the discharge, W in lbs/hr can be written,

$$W = 3600\ a\ v\ D = 3600aDK\sqrt{2g\ \frac{p}{D}} =$$

$$= 3600aK\sqrt{2g\ p\ D} = K_1\sqrt{p\ D} \qquad (92)$$

where,

a $=$ area of cross-section of the elbow through the plane of symmetry and $K_1 =$ elbow constant $= 3600\ aK\sqrt{2g\qquad\qquad}$; Thus, for our 3" elbow, $K_1 = 3600 \times .0516 \times .64\sqrt{64.4} = \mathbf{955}$, in the case of steam.

In the case of gases, where flow measurement in standard cubic feet/hr is usually wanted, the simplest way is to measure the weight of discharge and then divide the result by the constant density of a cubic foot of the gas under the standard conditions. The gas meter can then be made to read standard cubic feet directly, if the constant, K_1, of equation (92) is divided by the density of a standard cubic foot of the gas.

124. An inspection of equation (92) shows that an accurate flow meter must be able to measure the product of the two variables, pressure difference and density. If a definite relation exists between density, on the one hand, and absolute pressure, P, and temperature, T, on the other hand, as in the case of gases, for example, we may substitute for D in equation (92) its equivalent in terms of P and T.

$$P \cdot V = B\,T \qquad (98)$$ and

$$D = \frac{1}{V} \qquad (94)$$

from which, $D = \frac{1}{V} = \frac{P}{BT}$ (95)

and we may rewrite eq. (92) as

$$W = K_1 \sqrt{p\,D} = K_1 \sqrt{\frac{p\,P}{B\,T}} =$$

$$= K_2 \sqrt{\frac{p\,P}{T}} \qquad (96)$$

where the new constant K_2 involves the gas constant B; viz.,

$$K_2 = \frac{K_1}{\sqrt{B}} = \frac{3600\,aK\sqrt{2g}}{\sqrt{B}} = =$$

$$= \frac{28{,}900\ aK}{\sqrt{B}} \qquad (97)$$

Therefore, in the case of gases or in the case of any vapors whose densities vary with pressure and temperature in approximate accordance with the law,

$$PV = BT$$

e.g, superheated steam, the flow meter must be able to measure the combined product and ratio of the three variables, pressure difference, p, absolute pressure P, and absolute temperature, T.

125. This is accomplished in the case of Sargent flow meters by the use of electrical

resistances for measuring each of the 3 different variables, i.e., each of the latter is caused to vary an electrical resistance in direct proportion to its variations, Thus, we have,

$$p = C_1R_1 \qquad (98)$$

$$\mathbf{P} = C_2R_2 \qquad (99)$$

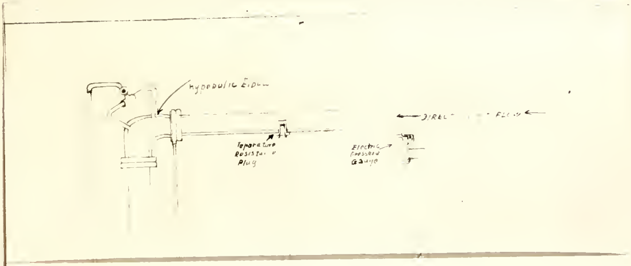
$$T = C_3R_3 \qquad (100)$$

where,

R_1 R_2 & R_3 are the three variable resistances and C_1 C_2 and C_3 are the corresponding constants of proportionality. Then use is made of the unique property of the Wheatstone bridge circuit viz.,that when the galvabometer reads zero, the 4 resistances of the circuit bear to each other the simple relation,

$$\frac{R}{R_1} = \frac{R_2}{R_3} \qquad (101)$$

(see Fig. 19)



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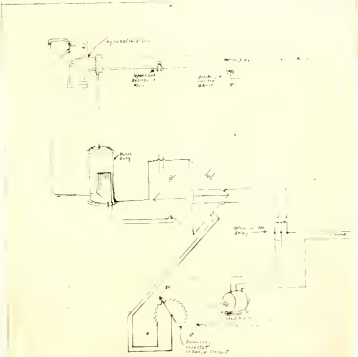

DIAGRAM OF ELECTRIC CIRCUIT

SARGENT FLOW METERS

or
$$R = \frac{R_1 R_2}{R_3} \qquad (102)$$

where R is the balancing unknown resistance of the circuit.

Since the expression

$$\frac{p\,P}{T}$$

involved in the flow formula,

$$W = K_2 \sqrt{\frac{p\,P}{T}} \qquad \text{(see eq. 96)}$$

is proportional to the corresponding resistance expression,

$$\frac{R_1\,R_2}{R_3}$$

viz.,

$$\frac{pP}{T} = \frac{C_1C_2}{C_3}\,\frac{R_1R_2}{R_3} \qquad (103)$$

(from EQ. 98,99,100),

it is evident that the unknown resistance, R, is also proportional to the expression

$\frac{pP}{T}$; viz.,

$$R = \frac{R_1R_2}{R_1} = \frac{C_3}{C1\;C_2}\,\frac{pP}{T} \qquad (103)$$

or the following simple relation exists between R and the rate of discharge, W:

$$W = K2\sqrt{\frac{pP}{T}} = K_2\sqrt{\frac{C_1C}{C_3}\qquad R} =$$

$$= K_3\sqrt{R} \qquad (104)$$

where the constant of proportionality, K_3, is evidently,

$$K_3 = K_2\sqrt{\frac{c_1 c_2}{c_3}} = \frac{28{,}900\,a\,K}{\sqrt{B}}\sqrt{\frac{c_1 c_2}{c_3}} \qquad (105)$$

126. For a full description of the new Sargent flow meters, the reader us advised to procure a copy of the report sent to the Fuild Meters Committee of the A. S. M. E. at a recent date; only a brief description of these meters is given below.

127. Fig. 19 is a diagramatic outline of the nature of the Wheatstone bridge circuit, as used for flow measurement. The pressure difference p is transmitted from the hyperbolic elbow or other metering device to the so-called "meter body" which is of the inclined mercury manometer type. The inclined tube is made of bakalite and has about one hundred metal contacts projected inside the tube and affected by variations of mercury level. Resistance coils are wound around the bakalite tube between the different contacts. By introducing the higher pressure on the inclined side of the manometers, the mercury level is lowered by the pressure difference and the effect-

ive resistance is increased in direct proportion to the movement of the mercury level or to the increase of pressure difference; i.e., we get,

$$p = C_1 R_1 \text{ (see eq.98)}$$

128. An electrical pressure gauge of the closed and mercury U-tube compressed air type is used for measuring absolute pressure. The mercury movement is caused to vary a resistance in accordance with the particular contact the mercury reaches. By properly distributing the resistances between the different contacts, the total resistance for any mercury level may be made any desired function of the absolute pressure. In case of gases and superheated steam, the pressure resistance is made directly proportional to absolute pressure i.e.,

$$P = C_2 R_2 \quad \text{(see eq. 99)}$$

In the case of wet or saturated steam for which, with constant known quality, the density depends upon absolute pressure alone, the pressure resistance is calibrated to be directly proportional to the density corresponding to the absolute pressure, as given by steam tables; i.e. the correction for density is as accurate as the steam tables can make it, and we have,

$$D = C_4 R_2 \quad (106)$$

129. To correct for temperature changes, use is made of the variation of resistance with temperature. If a resistance material could be chosen with such a resistance temperature co-efficient that the total resistance R_3 for any given absolute temperature T is directly proportional to the latter, we should have,

$$T = C_3 R_3 \quad \text{(see eq. 100)}$$

It can be shown that, if the temperature co-efficient a is referred to temperature, t_o, i.e., if

$$R = R_o \left[1 + a \, (t - t_o)\right], \qquad (107)$$

the desired value of this co-efficient is

$$a = \frac{1}{460 + t_o} = \frac{1}{T_o} \qquad (108)$$

e.g., in the case of air at room temperatures, let, $t_o = 60^\circ$ F. Then

$$a = \frac{1}{460 + 60} = \frac{1}{520} = .001925 \qquad (108a)$$

Thus, by substituting the expression for a as given by equation (108) into equation (107) we get,

$$R = R_o \left(1 + \frac{\cancel{1} + t - to}{460 + to}\right) = Ro \left(\frac{460 + to + t - to}{460 + to}\right) =$$

$$Ro \, \frac{(460 + t)}{460 + to} = \frac{Ro \quad T}{To} \qquad (109)$$

or the variable resistance, R, is directly proportional

to the absolute temperature, T, as was previously stated.

130. We use, for temperature correction, copper wire in series with a wire of some other material, which is practically unaffected by the temperature, such as constantan sometimes called Advance wire. The copper wire is wound into the form of a special coil, and the latter is encased into a metal plug and screwed into the pipe, where the flow is to be measured. The Advance wire coil is kept outside of the pipe at any desired distance from the latter, usually in the recording instruments case. The resultant temperature co-efficient of the copper and advance wire coils is less than that of copper,
since the Advance wire coil,

$$\left(\begin{aligned} a &= .004\ {}^{\circ}C = \\ &= .00222/{}^{\circ}F, \text{at room temperature} \end{aligned} \right)$$

has no temperature co-efficient. By proper proportioning the ratio between the copper and Advance wire resistances, any desired temperature co-efficient may be obtained. Thus for $a = .001925/{}^{\circ}F$, we use the ratio

$$C = \frac{R_{\text{Advance}}}{R_{\text{Copper}}} = \frac{a_{\text{Copper}}}{a} - 1 =$$

$$= \frac{.002220}{.001925} - 1 = 1.155 - 1 = .155$$

between the resistance of the Advance wire and that of the copper coil. The resultant coefficient can then be shown to be equal to $a = .001925/^\circ F$. Thus, at the reference temperature, to, for each ohm of copper resistance we have

$$1 + .155 = 1.155 \text{ total resistance}$$

At any higher temperature, t, the resistance becomes,

$$R = 1\left[1 + .00222\,(t - t_0)\right] + .155 =$$

$$= 1.155\left[1 + a\,(t - t_0)\right]$$

or

$$1.155 = \frac{1\left[1 + .00222\,(t - t_0)\right] + .155}{1 + a\,(t - t_0)}$$

$$= \frac{1.155 + .00222\,(t - t_0)}{1 + a\,(t - t_0)}$$

or

$$1.155 + 1.155\,a\,(t - t_0) = 1.155 + .00222\,(t - t_0)$$

or

$$a = \frac{.00222}{1.155} = .001925$$

as it should. The temperature resistance plug is shown inserted into the pipe of Fig. 19, and its resistance bears the following relation to absolute temperature,

$$T = C_3R_3 \text{ (See eq. 100)}$$

where, according to equation (109)

$$C_3 = \frac{T_o}{R_c} = \frac{(460 + t_o)}{(C + 1)\,R \text{ copper}} \quad (110)$$

$$\left(C = \frac{\text{RAdvance}}{\text{Rcopper}} \qquad \text{at } t_0\ {}^{\circ}\text{F}\right)$$

131. The balancing resistance, *R* , is a part of the recording apparatus of which only the indicator is shown diagramatically in Fig. 19, the graphic and integrating portions being ommitted. This rheostat is automatically adjusted in synchromism with variations of any of the three variable resistances i.e., meter body, pressure gauge or temperature plug resistances, by means of the galvonometer relay and the control motor shown in the figure. The galvonometer relay is a newly patented device which acts like an automatic switch starting the motor in either direction, when the galvonometer pointer deflects from its zero position in the corresponding direction and promptly stopping the motor, when the pointer returns to its neutral position. Its possible applications are innumerable, and its use for flow or any other metering devices is only one of its applications.

132. The motor is a small one of the Universal series type, which can be used either with direct or alternating currents. The comparatively large power of the motor as against that of ordinary instruments, whether mechanical or electrical, makes the recording apparatus unusually rugged and positive

in its action. The electric power consumed, however, is less than 25 watts for the motor, while for the Wheatstone bridge circuit the power used is less than 3 watts, the current of the circuit being less than .1 of an ampere and the voltage used being between 20 and 30 volts. The comparatively high voltage insures good electrical contact between the nickel projections and the mercury, while the comparatively small current is still large enough to make the galvonometer rugged. In fact a millivoltmeter could be used as a milliammeter in place of the galvanometer if provided with the patented relay attachment.

133. The accuracy of the Wheatstone bridge method of measuring resistance is proverbial. As a zero method, it is unaffected by voltage variations, there being no current flowing through the galvanometer when the bridge is balanced. In addition,the fact, that the Wheatstone bridge makes it possible to measure the product, ratio, ~~ratio~~ or combinations of both of several variables is of great practical importance.

134. To summarize: (1) The reliability of the hyperbolic elbow for producting a difference of pressure which has a definite predeterminable relation to the average velocity, (2) the accuracy of the resistance method of measuring variation

of mercury level, produced by a difference of pressure, and finally, (3) the possibility of correcting for variations of absolute pressure, temperature, density etc., by means of the Wheatstone bridge circuit - all these important points open a new era in flow measurement accuracy.

135. It is hoped that the method used in this investigation for analyzing plane stream flow, will be found useful for additional similar investigations. It is also hoped that the scientific world will find it possible to continue the research work along these lines, corrobating or modifying the results so far found. The Sargent Steam Meter Company wishes to offer any assistance within its power for such investigations and will for such purposes furnish hyperbolic elbows up to 12" size at cost.

136. The writer desires to acknowledge his indebtedness to Prof. Gebhardt and the rest of the Faculty of the Armour Institute of Technology for making it possible to carry out this investigation as well as for encouragement and general co-operation. The writer also wishes to credit Prof. Lunn, Professor of Mathematical Physics, University of Chicago, for valuable suggestions and references on the subjects of hydrodynamics and differential equations.